

# POSITIVE PARTITION RELATIONS FOR $P_\kappa(\lambda)$

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**Abstract.** Let  $\kappa$  a regular uncountable cardinal and  $\lambda$  a cardinal  $> \kappa$ , and suppose  $\lambda^{<\kappa}$  is less than the covering number for category  $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ . Then (a)  $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+, \omega + 1)^2$ , (b)  $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_{\kappa^+}^2$  if  $\kappa$  is a limit cardinal, and (c)  $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$  if  $\kappa$  is weakly compact.

## 0. Introduction

Let  $\kappa$  be a weakly compact cardinal. Then  $\kappa \rightarrow (\kappa)^2$  and more generally for any cardinal  $\lambda \geq \kappa$ ,  $\{P_\kappa(\lambda)\} \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$  ([M4]), which means that for any  $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$ , there is  $A \subseteq P_\kappa(\lambda)$  such that  $A$  does not belong to  $I_{\kappa,\lambda}$  (the ideal of noncofinal subsets of  $P_\kappa(\lambda)$ ) and  $F$  is constant on

$$\{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\}.$$

Now if  $J$  is the ideal of noncofinal subsets of  $\kappa$ , then  $J^+ \rightarrow (J^+)^2$  since  $(A, <)$  is isomorphic to  $(\kappa, <)$  for any  $A \in J^+$ . So it is natural to ask whether  $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$  for every  $\lambda > \kappa$ . It turns out that the answer is negative. This is not surprising since it is well-known that some members of  $I_{\kappa,\lambda}^+$  may be quite different from  $P_\kappa(\lambda)$ . To give an example, if the GCH holds and  $\lambda$  is the successor of a cardinal of cofinality  $< \kappa$ , then  $\overline{\text{cof}}(I_{\kappa,\lambda} \mid A) < \overline{\text{cof}}(I_{\kappa,\lambda})$  for some  $A \in I_{\kappa,\lambda}^+$  ([MPéS2]). We prove that  $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$  if and only if  $\lambda^{<\kappa}$  is less than  $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$  (a generalization of the covering number for category  $\mathbf{cov}(\mathbf{M})$ ).

Let  $\kappa$  be an arbitrary regular uncountable cardinal. Dushnik and Miller [DMi] established that  $\kappa \rightarrow (\kappa, \omega)^2$ . This was improved to  $\kappa \rightarrow (\kappa, \omega + 1)^2$  by Erdős and Rado [ER]. The Erdős-Rado result generalizes ([M3]) : for every cardinal  $\lambda \geq \kappa$ ,  $\{P_\kappa(\lambda)\} \xrightarrow{\kappa} (I_{\kappa,\lambda}^+, \omega + 1)^2$  (i.e. for any  $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$ , there is either  $A \in I_{\kappa,\lambda}^+$  such that  $F$  is identically 0 on

$$\{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\},$$

or  $a_0, a_1, \dots, a_\omega$  in  $P_\kappa(\lambda)$  such that  $a_0 \subset a_1 \subset \dots \subset a_\omega$ ,  $\cup(a_0 \cap \kappa) < \cup(a_1 \cap \kappa) < \dots < \cup(a_\omega \cap \kappa)$  and  $F$  is identically 1 on  $\{(\cup(a_n \cap \kappa), a_q) : n < q \leq \omega\}$ ). Here we show that  $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+, \omega + 1)^2$  if  $\lambda^{<\kappa}$  is less than  $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ . In the other direction we prove that  $I_{\kappa,\lambda}^+ \not\xrightarrow{\kappa} (I_{\kappa,\lambda}^+, 3)^2$  if  $\lambda$  is greater than or equal to  $\mathfrak{d}_\kappa$  (or even  $\bar{\mathfrak{d}}_\kappa$ ).

It is a result of [M5] that  $\{P_\kappa(\lambda)\} \not\xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_\lambda^2$  for any  $\lambda > \kappa$  if  $\kappa$  is a successor cardinal such that  $\kappa \not\rightarrow [\kappa]_\kappa^2$ . In contrast to this, we show that  $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_{\kappa^+}^2$  if  $\kappa$  is a limit cardinal and  $\lambda$  a cardinal  $> \kappa$  with  $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ . It is also shown that  $I_{\kappa,\lambda}^+ \not\xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_\lambda^2$  if  $\lambda \geq \bar{\mathfrak{d}}_\kappa$ .

**Throughout the remainder of this paper  $\kappa$  will denote a regular uncountable cardinal and  $\lambda$  a cardinal  $> \kappa$ .**

The paper is organized as follows. Section 1 reviews a number of standard definitions concerning ideals on  $\kappa$  and  $P_\kappa(\lambda)$ . Sections 2-7 give results about combinatorics on  $\kappa$  that are needed for our study of  $P_\kappa(\lambda)$ . Sections 2 and 3 review some facts concerning, respectively, the dominating number  $\mathfrak{d}_\kappa$  and the

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covering number for category  $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ . Section 4 deals with the problem of determining the value of the inequality number  $\mathfrak{U}_\kappa$  in the case where  $\kappa$  is a successor cardinal. In Section 5 we show that if  $2^{<\kappa} = \kappa$  and  $\mathfrak{U}_\kappa < \kappa^{+\omega}$ , then  $\mathfrak{U}_\kappa = \mathbf{non}_\kappa$  (weakly selective). Sections 6 and 7 review some material concerning, respectively, the unbalanced partition relation  $J^+ \longrightarrow (J^+, \rho)^2$  and the square bracket partition relation  $J^+ \longrightarrow [J^+]^2_\rho$ .

Sections 8-15 are concerned with combinatorial properties of ideals on  $P_\kappa(\lambda)$ . Section 8 gives two characterizations of  $\mathfrak{D}_{\kappa,\lambda}^\kappa$ : one as the least cofinality of any  $\kappa$ -complete fine ideal on  $P_\kappa(\lambda)$  that is not a weak  $\pi$ -point, and the other as the least cofinality of any  $\kappa$ -complete fine ideal on  $P_\kappa(\lambda)$  that admits a maximal almost disjoint family of size  $\kappa$ . In Section 9 we show that any  $\kappa$ -complete fine ideal on  $P_\kappa(\lambda)$  with cofinality  $< \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$  is a weak  $\chi$ -point. Conversely if  $\kappa$  is inaccessible and  $I_{\kappa,\lambda}$  is a weak  $\chi$ -point, then  $\text{cof}(I_{\kappa,\lambda}) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ . Sections 10-13 deal with unbalanced partition relations. Given an infinite cardinal  $\theta \leq \kappa$  such that  $\kappa \rightarrow (\kappa, \theta)^2$ , we show that (a)  $u(\kappa, \lambda) \cdot \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \theta)^2)$  is the least cofinality of any  $\kappa$ -complete fine ideal  $H$  on  $P_\kappa(\lambda)$  such that  $H^+ \not\overset{\kappa}{\longrightarrow} (H^+; \theta)^2$ , (b) If  $H$  is a  $\kappa$ -complete fine ideal on  $P_\kappa(\lambda)$  with  $\text{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$  (respectively,  $\text{cof}(H) < \mathbf{non}_\kappa$  (weakly selective)), then  $H^+ \longrightarrow (H^+, \theta)^2$  (respectively,  $H^+ \overset{\kappa}{\longrightarrow} (H^+, \theta)^2$ ), and (c) Conversely, if  $\theta = \kappa$  and  $I_{\kappa,\lambda}^+ \overset{\kappa}{\longrightarrow} (H^+, \theta)^2$ , then  $\text{cof}(I_{\kappa,\lambda}) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ . The last two sections are concerned with square bracket partition relations. We show that if  $\kappa$  is a limit cardinal, then  $H^+ \overset{\kappa}{\longrightarrow} [H^+]^2_{\kappa^+}$  (respectively,  $H^+ \overset{\kappa}{\longrightarrow} [H^+]^2_\kappa$ ) for every ideal  $H$  on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$  (respectively,  $\text{cof}(H) < \mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_\kappa)$ ). In the other direction,  $\lambda \geq \overline{\mathfrak{D}}_\kappa$  implies that  $I_{\kappa,\lambda}^+ \not\overset{\kappa}{\longrightarrow} [I_{\kappa,\lambda}^+]^2_\lambda$  (and  $I_{\kappa,\lambda}^+ \not\overset{\kappa}{\longrightarrow} [I_{\kappa,\lambda}^+]^2_\kappa$  if  $\kappa$  is a limit cardinal such that  $2^{<\kappa} = \kappa$ ).

## 1. Ideals

In this section we review some standard definitions and a few basic facts concerning ideals on  $\kappa$  and  $P_\kappa(\lambda)$ .

Given a cardinal  $\mu$  and a set  $A$ , let  $P_\mu(A) = \{a \subseteq A : |a| < \mu\}$ .

Given an infinite set  $S$ , an *ideal on  $S$*  is a collection  $K$  of subsets of  $S$  such that (i)  $\{s\} \in K$  for every  $s \in S$ , (ii)  $P(A) \subseteq K$  for every  $A \in K$ , (iii)  $A \cup B \in K$  whenever  $A, B \in K$ , and (iv)  $S \notin K$ .

Given an ideal  $K$  on  $S$ , let  $K^+ = P(S) - K$  and  $K \restriction A = \{B \subseteq S : B \cap A \in K\}$  for  $A \in K^+$ .  $\text{sat}(K)$  is the least cardinal  $\tau$  with the property that for every  $Y \subseteq K^+$  with  $|Y| = \tau$ , there exist  $A, B \in Y$  such that  $A \neq B$  and  $A \cap B \in K^+$ .

$\text{cof}(K)$  is the least cardinality of any  $X \subseteq K$  such that  $K = \bigcup_{A \in X} P(A)$ .  $K$  is  $\kappa$ -complete if  $\bigcup X \in K$  for every  $X \in P_\kappa(K)$ . Assuming that  $K$  is  $\kappa$ -complete and  $\bigcup Y \in K^+$  for some  $Y \subseteq K$  with  $|Y| = \kappa$ ,  $\overline{\text{cof}}(K)$  is the least cardinality of any  $X \subseteq K$  such that  $K = \bigcup \{P(\cup x) : x \in P_\kappa(X)\}$ .

**We adopt the convention that the phrase “ideal on  $\kappa$ ” means “ $\kappa$ -complete ideal on  $\kappa$ ”.**

Note that the smallest ideal on  $\kappa$  is  $P_\kappa(\kappa)$ .

Given two sets  $A$  and  $B$  and  $f \in {}^A B$ ,  $f$  is *regressive* if  $f(a) \in a$  for all  $a \in A$ .

An ideal  $J$  on  $\kappa$  is *normal* if given  $A \in J^+$  and a regressive  $f \in {}^A \kappa$ , there is  $B \in J^+ \cap P(A)$  such that  $f$  is constant on  $B$ .

$NS_\kappa$  denotes the nonstationary ideal on  $\kappa$ .

$\kappa$  is *inaccessible* if  $2^\mu < \kappa$  for every cardinal  $\mu < \kappa$ .

Let  $[A]^2 = \{(\alpha, \beta) \in A \times A : \alpha < \beta\}$  for any  $A \subseteq \kappa$ . Given an ordinal  $\alpha \geq 2$ ,  $\kappa \rightarrow (\kappa, \alpha)^2$  means that for every  $f : [\kappa]^2 \rightarrow \alpha$ , there is  $A \subseteq \kappa$  such that either  $A$  has order type  $\kappa$  and  $f$  is identically 0 on  $[A]^2$ , or  $A$  has order type  $\alpha$  and  $f$  is identically 1 on  $[A]^2$ . The negation of this and other partition relations is indicated by crossing the arrow.  $\kappa \rightarrow (\kappa)^2$  means that  $\kappa \rightarrow (\kappa, \kappa)^2$ .

$\kappa$  is *weakly compact* if  $\kappa \rightarrow (\kappa)^2$ .

If  $\kappa$  is weakly compact, then it is inaccessible (see e.g. Proposition 4.4 in [Ka]).

An ideal  $J$  on  $\kappa$  is a *weak  $P$ -point* if given  $A \in J^+$  and  $f \in {}^A\kappa$  with  $\{f^{-1}(\{\alpha\}) : \alpha < \kappa\} \subseteq J$ , there is  $B \in J^+ \cap P(A)$  such that  $f$  is  $< \kappa$ -to-one on  $B$ .  $J$  is a *local  $Q$ -point* if given  $g \in {}^\kappa\kappa$ , there is  $B \in J^+$  such that  $g(\alpha) < \beta$  for any  $(\alpha, \beta) \in [B]^2$ .  $J$  is a *weak  $Q$ -point* if  $J \restriction A$  is a local  $Q$ -point for every  $A \in J^+$ .

It is well-known (see [M1] for a proof) that an ideal  $J$  on  $\kappa$  is a weak  $Q$ -point if and only if given  $A \in J^+$  and a  $< \kappa$ -to-one  $f : A \rightarrow \kappa$ , there is  $B \in J^+ \cap P(A)$  such that  $f$  is one-to-one on  $B$ .

An ideal  $J$  on  $\kappa$  is *weakly selective* if it is both a weak  $P$ -point and a weak  $Q$ -point.

Given a cardinal  $\rho$  with  $2 \leq \rho \leq \kappa$  and an ideal  $J$  on  $\kappa$ ,  $J^+ \rightarrow [J^+]_\rho^2$  means that for every  $A \in J^+$  and every  $f : [A]^2 \rightarrow \rho$ , there is  $B \in J^+ \cap P(A)$  such that  $f''[B]^2 \neq \rho$ .  $\kappa \rightarrow [\kappa]_\rho^2$  means that  $(P_\kappa(\kappa))^+ \rightarrow [(P_\kappa(\kappa))^+]_\rho^2$ .

Note that  $\kappa \rightarrow [\kappa]_2^2$  if and only if  $\kappa \rightarrow (\kappa)^2$ .

Let  $P$  be a property such that at least one ideal on  $\kappa$  does not satisfy  $P$ . Then  $\mathbf{non}_\kappa(P)$  (respectively,  $\overline{\mathbf{non}}_\kappa(P)$ ) denotes the least cardinal  $\tau$  for which one can find an ideal  $J$  on  $\kappa$  such that  $\text{cof}(J) = \tau$  (respectively,  $\overline{\text{cof}}(J) = \tau$ ) and  $J$  does not satisfy  $P$ .

Notice that  $\lambda^{<\kappa} < \overline{\mathbf{non}}_\kappa(P)$  if and only if  $\lambda^{<\kappa} < \mathbf{non}_\kappa(P)$ .

$I_{\kappa, \lambda}$  denotes the set of all  $A \subseteq P_\kappa(\lambda)$  such that  $A \cap \{b \in P_\kappa(\lambda) : a \subseteq b\} = \emptyset$  for some  $a \in P_\kappa(\lambda)$ . An ideal  $H$  on  $P_\kappa(\lambda)$  is *fine* if  $I_{\kappa, \lambda} \subseteq H$ .

**We adopt the convention that the phrase “ideal on  $P_\kappa(\lambda)$ ” means “ $\kappa$ -complete fine ideal on  $P_\kappa(\lambda)$ ”.**

Note that  $I_{\kappa, \lambda}$  is the smallest ideal on  $P_\kappa(\lambda)$ .

$u(\kappa, \lambda)$  denotes the least cardinality of any  $A \in I_{\kappa, \lambda}^+$ .

The following facts are well-known (see e.g. [MPéS1]) : (1)  $u(\kappa, \lambda) \geq \lambda$  ; (2)  $\lambda^{<\kappa} = 2^{<\kappa} \cdot u(\kappa, \lambda)$  ; (3)  $u(\kappa, \lambda) = \text{cof}(I_{\kappa, \lambda} \restriction A)$  for every  $A \in I_{\kappa, \lambda}^+$  ; (4)  $u(\kappa, \kappa^{+n}) = \kappa^{+n}$  whenever  $0 < n < \omega$ .

$\mathcal{K}(\kappa, \lambda)$  denotes the set of all cardinals  $\sigma \geq \lambda$  with the property that there is  $T \subseteq P_\kappa(\lambda)$  such that  $|T| = \sigma$  and  $|T \cap P(a)| < \kappa$  for every  $a \in P_\kappa(\lambda)$ .

It is simple to see that  $\sigma \leq u(\kappa, \lambda)$  for every  $\sigma \in \mathcal{K}(\kappa, \lambda)$ . Notice that  $\lambda \in \mathcal{K}(\kappa, \lambda)$ . More generally, if  $\tau$  is an infinite cardinal  $\leq \kappa$  such that  $|P_\tau(\nu)| < \kappa$  for every infinite cardinal  $\nu < \kappa$ , then  $\lambda^{<\tau} \in \mathcal{K}(\kappa, \lambda)$ . It follows that  $\lambda^{<\kappa} \in \mathcal{K}(\kappa, \lambda)$  if  $\kappa$  is inaccessible. It can be shown (see Remark 11.4 in [To 2] and Theorem 4.1 in [CFMag]) that  $\lambda^+ \in \mathcal{K}(\kappa, \lambda)$  if  $\square_\kappa^*$  holds and  $\text{cf}(\lambda) < \kappa$ .

An ideal  $H$  on  $P_\kappa(\lambda)$  is  $\kappa$ -*normal* if given  $A \in H^+$  and a regressive  $f \in {}^A\kappa$ , there is  $B \in H^+ \cap P(A)$  such that  $f$  is constant on  $B$ . The smallest  $\kappa$ -normal ideal on  $P_\kappa(\lambda)$  is denoted by  $NS_{\kappa, \lambda}^\kappa$ .

## 2. Domination

In this section we recall some characterizations of the dominating number  $\mathfrak{d}_\kappa$ .

**Definition.**  $\mathfrak{d}_\kappa$  is the least cardinality of any  $X \subseteq {}^\kappa\kappa$  with the property that for every  $g \in {}^\kappa\kappa$ , there is  $f \in X$  such that  $g(\alpha) < f(\alpha)$  for all  $\alpha < \kappa$ .

$\overline{\mathfrak{d}}_\kappa$  is the least cardinality of any  $X \subseteq {}^\kappa\kappa$  with the property that for every  $g \in {}^\kappa\kappa$ , there is  $x \in P_\kappa(X)$  such that  $g(\alpha) < \bigcup_{f \in x} f(\alpha)$  for all  $\alpha < \kappa$ .

### PROPOSITION 2.1.

(i) ([L1])  $\mathfrak{d}_\kappa = \text{cof}(NS_\kappa)$ .

(ii) ([MRoS])  $\overline{\mathfrak{d}}_\kappa = \overline{\text{cof}}(NS_\kappa)$ .

**Definition.** Given an ideal  $J$  on  $\kappa$ ,  $\mathcal{M}_J^{\geq \kappa}$  is the set of all  $Q \subseteq J^+$  such that (i)  $|Q| \geq \kappa$ , (ii)  $A \cap B \in J$  for all  $A, B \in Q$  with  $A \neq B$ , and (iii) for every  $C \in J^+$ , there is  $A \in Q$  with  $A \cap C \in J^+$ .

$\mathfrak{a}_J$  is the least cardinality of any member of  $\mathcal{M}_J^{\geq \kappa}$  if  $\mathcal{M}_J^{\geq \kappa} \neq \emptyset$ , and  $(2^\kappa)^+$  otherwise.

**THEOREM 2.2.** ([Laf], [MP2])  $\mathfrak{d}_\kappa = \mathbf{non}_\kappa(\mathfrak{a}_J > \kappa) = \mathbf{non}_\kappa(\text{weak } P\text{-point})$ .

**PROPOSITION 2.3.**  $\overline{\mathfrak{d}}_\kappa \geq \overline{\mathbf{non}}_\kappa(\mathfrak{a}_J > \kappa) \geq \overline{\mathbf{non}}_\kappa(\text{weak } P\text{-point})$ .

**Proof.** The first inequality follows from Proposition 2.1 (ii) since  $\mathfrak{a}_{NS_\kappa} = \kappa$  ([MP2]). To prove the second inequality, argue as for Lemma 8.5 below.  $\square$

**QUESTION.** Is it consistent that  $\overline{\mathfrak{d}}_\kappa > \overline{\mathbf{non}}_\kappa(\text{weak } P\text{-point})$ ?

## 3. Covering for category

Throughout this section  $\nu$  will denote a fixed regular infinite cardinal.

We will review some basic facts concerning the covering number  $\mathbf{cov}(\mathbf{M}_{\nu,\nu})$ .

**Definition.** Suppose  $\rho$  is a cardinal  $\geq \nu$ .

Let  $F_n(\rho, 2, \nu) = \cup \{a2 : a \in P_\nu(\rho)\}$ .  $F_n(\rho, 2, \nu)$  is ordered by  $p \leq q$  if and only if  $q \subseteq p$ .

${}^\rho 2$  is endowed with the topology obtained by taking as basic open sets  $\phi$  and  $O_s^\rho$  for  $s \in F_n(\rho, 2, \nu)$ , where  $O_s^\rho = \{f \in {}^\rho 2 : s \subseteq f\}$ .

$\mathbf{M}_{\nu,\rho}$  is the set of all  $W \subseteq {}^\rho 2$  such that  $W \cap (\cap X) = \emptyset$  for some collection  $X$  of dense open subsets of  ${}^\rho 2$  with  $0 < |X| \leq \nu$ .

$\mathbf{cov}(\mathbf{M}_{\nu,\rho})$  is the least cardinality of any  $Y \subseteq \mathbf{M}_{\nu,\rho}$  such that  ${}^\rho 2 = \cup Y$ .

### PROPOSITION 3.1.

(i) ([L2],[Mil2])  $\mathbf{cov}(\mathbf{M}_{\nu,\rho}) \geq \nu^+$  for every cardinal  $\rho \geq \nu$ .

(ii) ([L2],[Mil2]) Suppose that  $\rho$  and  $\mu$  are two cardinals such that  $\nu \leq \mu \leq \rho$ . Then  $\mathbf{cov}(\mathbf{M}_{\nu,\mu}) \geq \mathbf{cov}(\mathbf{M}_{\nu,\rho})$ .

(iii) ([L2]) Suppose  $2^{<\nu} > \nu$ . Then  $\mathbf{cov}(\mathbf{M}_{\nu,\nu}) = \nu^+$ .

**PROPOSITION 3.2.** Suppose that  $\rho$  is a cardinal  $> \nu$  and  $V \models 2^{<\nu} = \nu$ . Then setting  $P = \text{Fn}(\rho, 2, \nu)$  :

- (i) ([L2],[Mil2])  $V^P \models \mathbf{cov}(\mathbf{M}_{\nu,\rho}) \geq \rho$ .
- (ii) ([L2],[Mil2]) If  $cf(\rho) \leq \nu$ , then  $V^P \models \mathbf{cov}(\mathbf{M}_{\nu,\nu}) > \rho$ .
- (iii) Let  $\mu$  be any regular cardinal  $> \nu$ . Then  $(\mathfrak{d}_\mu)^{V^P} = (\mathfrak{d}_\mu)^V$  and  $(\bar{\mathfrak{d}}_\mu)^{V^P} \leq (\bar{\mathfrak{d}}_\mu)^V$ .

**Proof.** (iii) : The conclusion easily follows from the following observation : Suppose that  $\sigma$  is a cardinal  $> 0$  and  $F \in V^P$  is a function from  $\sigma \times \mu$  to  $\mu$ . Then by Lemma VII.6.8 of [K], there is  $H : \sigma \times \mu \rightarrow P_{\nu^+}(\mu)$  such that  $H \in V$  and  $F(\alpha, \beta) \in H(\alpha, \beta)$  (so  $F(\alpha, \beta) \leq \cup H(\alpha, \beta)$ ) for every  $(\alpha, \beta) \in \sigma \times \mu$ .  $\square$

**Remark.** It is not known whether it is consistent that  $cf(\mathbf{cov}(\mathbf{M}_{\nu,\nu})) \leq \nu$ .

#### 4. Unequality

Our main concern in this section is with the problem of evaluating the inequality number  $\mathfrak{U}_\kappa$  when  $\kappa$  is a successor cardinal.

**Definition.**  $\mathfrak{U}_\kappa$  (respectively,  $\mathfrak{U}'_\kappa$ ) is the least cardinality of any  $F \subseteq {}^\kappa\kappa$  with the property that for every  $g \in {}^\kappa\kappa$ , there is  $f \in F$  such that  $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} = \phi$  (respectively  $|\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}| < \kappa$ ).

The following is readily checked.

**PROPOSITION 4.1.**  $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) \leq \mathfrak{U}_\kappa \leq \mathfrak{d}_\kappa$ .

**Remark.** It is shown in [MRoS] that if  $V \models \text{GCH}$ , then there is a  $\kappa$ -complete  $\kappa^+$ -cc forcing notion  $P$  such that

$$V^P \models \text{"}\bar{\mathfrak{d}}_\kappa = \kappa^{+\omega} \text{ and } \mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) = 2^\kappa = \kappa^{+(\omega+1)}\text{"}.$$

For models where  $\mathfrak{d}_\kappa > \kappa^+$  see also [CS].

**PROPOSITION 4.2.**  $\mathfrak{U}_\kappa = \mathfrak{U}'_\kappa$ .

**Proof.** Fix  $F \subseteq {}^\kappa\kappa$  with the property that for every  $g \in {}^\kappa\kappa$ , there is  $f \in F$  such that

$$|\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}| < \kappa.$$

For  $f \in F$  and  $\gamma, \delta < \kappa$ , define  $f_{\gamma,\delta} \in {}^\kappa\kappa$  by :  $f_{\gamma,\delta}(\alpha) = f(\alpha)$  if  $\alpha \geq \gamma$ , and  $f_{\gamma,\delta}(\alpha) = \delta$  otherwise. Then for every  $g \in {}^\kappa\kappa$ , there are  $f \in F$  and  $\gamma, \delta < \kappa$  such that  $\{\alpha \in \kappa : f_{\gamma,\delta}(\alpha) = g(\alpha)\} = \phi$ .  $\square$

The following is due to Landver [L2].

**PROPOSITION 4.3.**  $cf(\mathfrak{U}_\kappa) > \kappa$ .

**Proof.** Suppose otherwise. Set  $\nu = cf(\mathfrak{U}_\kappa)$  and fix  $F \subseteq {}^\kappa\kappa$  so that  $|F| = \mathfrak{U}_\kappa$  and for every  $g \in {}^\kappa\kappa$ , there exists  $f \in F$  with  $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} = \phi$ . Let  $\langle F_\beta : \beta < \nu \rangle$  be such that (a)  $|F_\beta| < \mathfrak{U}_\kappa$  for any  $\beta$ , and (b)  $\bigcup_{\beta < \nu} F_\beta = F$ . Select  $A_\beta \subseteq \kappa$  for  $\beta < \nu$  so that (i)  $|A_\beta| = \kappa$  for every  $\beta < \nu$ , (ii)  $A_\beta \cap A_\gamma = \phi$  whenever

$\gamma < \beta < \nu$ , and (iii)  $\bigcup_{\beta < \nu} A_\beta = \kappa$ . For each  $\beta < \nu$ , there is  $g_\beta : A_\beta \rightarrow \kappa$  such that

$$\{\alpha \in A_\beta : (f \upharpoonright A_\beta)(\alpha) = g_\beta(\alpha)\} \neq \phi$$

for every  $f \in F_\beta$ . Set  $g = \bigcup_{\beta < \nu} g_\beta$ . Then clearly,  $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \neq \phi$  for all  $f \in F$ . This is a contradiction.  $\square$

We now turn our attention to the task of computing  $\mathfrak{U}_\kappa$ . We begin with the case when  $\kappa$  is a successor cardinal.

**THEOREM 4.4.** *Suppose  $\kappa$  is the successor of a regular infinite cardinal  $\nu$ . Then*

$$\mathfrak{U}_\kappa \geq \min(\mathfrak{D}_\kappa, \mathbf{cov}(\mathbf{M}_{\nu, \kappa})).$$

**Proof.** Fix  $F \subseteq {}^\kappa \kappa$  with  $0 < |F| < \min(\mathfrak{D}_\kappa, \mathbf{cov}(\mathbf{M}_{\nu, \kappa}))$ . Pick  $k : \kappa \rightarrow \kappa - \nu$  so that

$$|\{\alpha < \kappa : k(\alpha) > f(\alpha)\}| = \kappa$$

for every  $f \in F$ . Select a bijection  $j : \kappa \times \nu \rightarrow \kappa$  and a bijection  $i_\alpha : k(\alpha) \rightarrow \nu$  for each  $\alpha < \kappa$ . Given  $A \subseteq \kappa$  and  $t \in {}^A 2$ , define a partial function  $\bar{t}$  from  $\kappa$  to  $\kappa$  by stipulating that  $\bar{t}(\alpha) = \gamma$  if and only if (a)  $\gamma < k(\alpha)$ , (b)  $\{j(\alpha, \eta) : \eta < i_\alpha(\gamma)\} \subseteq t^{-1}(\{0\})$ , and (c)  $j(\alpha, i_\alpha(\gamma)) \in t^{-1}(\{1\})$ . For  $f \in F$ , let  $D_f$  be the set of all  $s \in Fn(\kappa, 2, \nu)$  such that there is  $\alpha \in \text{dom}(\bar{s})$  with  $k(\alpha) > f(\alpha)$  and  $\bar{s}(\alpha) = f(\alpha)$ . Clearly, each  $D_f$  is a dense subset of  $Fn(\kappa, 2, \nu)$ , so we can find  $g \in {}^\kappa 2$  with the property that for every  $f \in F$ , there is  $a \in P_\nu(\kappa)$  with  $g \restriction a \in D_f$ . Then

$$\{\alpha \in \text{dom}(\bar{g}) : \bar{g}(\alpha) = f(\alpha)\} \neq \emptyset$$

for every  $f \in F$ . □

**THEOREM 4.5.** *Suppose  $\kappa$  is a successor cardinal. Then  $\mathfrak{U}_\kappa \geq \bar{\mathfrak{D}}_\kappa$ .*

**Proof.** Fix  $F \subseteq {}^\kappa \kappa$  with  $0 < |F| < \bar{\mathfrak{D}}_\kappa$ . Set  $\kappa = \nu^+$ . Pick  $k : \kappa \rightarrow \kappa - \nu$  so that

$$|\{\alpha < \kappa : f(\alpha) < k(\alpha)\}| = \kappa$$

for every  $f \in F$ . For  $\alpha < \kappa$ , select a bijection  $\pi_\alpha : k(\alpha) \rightarrow \nu$ . Given  $f \in F$ , there exists  $i_f \in \nu$  such that the set

$$A_f = \{\alpha < \kappa : f(\alpha) < k(\alpha) \text{ and } \pi_\alpha(f(\alpha)) = i_f\}$$

has size  $\kappa$ . Define  $g_f \in {}^\kappa \kappa$  by

$$g_f(\beta) = \text{least } \alpha \in A_f \text{ such that } \alpha \geq \beta.$$

It is shown in [MRoS] that  $\bar{\mathfrak{D}}_\kappa$  is the least cardinality of any  $X \subseteq {}^\kappa \kappa$  with the property that for every  $h \in {}^\kappa \kappa$ , there is  $x \in P_\kappa(X)$  such that the set  $\{\beta < \kappa : h(\beta) \geq \bigcup_{f \in x} f(\beta)\}$  is nonstationary in  $\kappa$ . Hence there is  $h \in {}^\kappa \kappa$  such that the set

$$B_x = \{\beta < \kappa : h(\beta) \geq \bigcup_{f \in x} g_f(\beta)\}$$

is stationary in  $\kappa$  for every  $x \in P_\kappa(F)$ .

Define  $J \subseteq P(\kappa)$  by  $D \in J$  if and only if there is  $x \in P_\kappa(F)$  such that  $D \cap B_x \in NS_\kappa$ . Then  $J$  is an ideal on  $\kappa$ . Since  $\text{sat}(J) > \nu$  by a result of Ulam (see [Ka], 16.3), there exist pairwise disjoint  $D_i \in J^+$  for  $i < \nu$  with  $\bigcup_{i < \nu} D_i = \kappa$ .

Let  $C$  be the set of all infinite limit ordinals  $\delta < \kappa$  such that  $h(\xi) < \delta$  for every  $\xi < \delta$ . Then  $C$  is a closed unbounded subset of  $\kappa$ . Define  $t \in {}^\kappa \kappa$  so that for every  $\eta < \kappa$ ,  $t(\eta) < k(\eta)$  and  $c_\eta \in D_{\pi_\eta(t(\eta))}$ , where  $c_\eta = \cup(C \cap \eta)$ .

Now fix  $f \in F$ . Pick  $\zeta \in D_{i_f} \cap C \cap B_{\{f\}}$  and set  $\eta = g_f(\zeta)$ . Notice that  $\zeta \leq \eta$  by the definition of  $g_f$ . Also,  $\eta \leq h(\zeta)$  since  $\zeta \in B_{\{f\}}$ . Hence  $c_\eta = \zeta$  by the definition of  $C$  and the fact that  $\zeta \in C$ . It now follows from the definition of  $t$  and the fact that  $\zeta \in D_{i_f}$  that  $\pi_\eta(t(\eta)) = i_f$ . On the other hand,  $\eta \in A_f$  since  $\eta = g_f(\zeta)$ , so  $f(\eta) < k(\eta)$  and  $\pi_\eta(f(\eta)) = i_f$ . Thus  $t(\eta) = f(\eta)$ .  $\square$

**Remark.** It follows from Proposition 4.1 and Theorem 4.5 that  $\mathfrak{U}_\kappa = \mathfrak{d}_\kappa$  if  $\kappa$  is a successor cardinal and  $\mathfrak{d}_\kappa < \kappa^{+\omega}$ .

**THEOREM 4.6.** Suppose that  $\kappa$  is a successor cardinal and  $2^{<\kappa} = \kappa$ . Then  $\mathfrak{U}_\kappa = \mathfrak{d}_\kappa$ .

**Proof.** By Proposition 4.1 it suffices to prove that  $\mathfrak{U}_\kappa \geq \mathfrak{d}_\kappa$ . Set  $\kappa = \nu^+$  and select a one-to-one

$$j : \bigcup_{\alpha < \kappa} {}^{[\alpha, \alpha + \nu)} \kappa \longrightarrow \kappa.$$

Now fix  $F \subseteq {}^\kappa \kappa$  with  $0 < |F| < \mathfrak{d}_\kappa$ . Select  $g \in {}^\kappa \kappa$  so that for every  $f \in F$ , there is  $\beta_f < \kappa$  with

$$j(f \upharpoonright [\beta_f, \beta_f + \nu)) < g(\beta_f).$$

Let  $C$  be the set of all  $\gamma < \kappa$  such that  $\beta + \nu < \gamma$  and  $g(\beta) < \gamma$  for every  $\beta < \gamma$ . Then  $C$  is a closed unbounded subset of  $\kappa$ . Let  $\langle \gamma_\delta : \delta < \kappa \rangle$  be the increasing enumeration of  $C$ . For  $\delta < \kappa$ , set

$$W_\delta = \left\{ t \in \bigcup_{\gamma_\delta \leq \alpha < \gamma_{\delta+1}} {}^{[\alpha, \alpha + \nu)} \kappa : j(t) < \gamma_{\delta+1} \right\}.$$

Then define  $k_\delta \in {}^{[\gamma_\delta, \gamma_{\delta+1})} \kappa$  so that for every  $t \in W_\delta$ , there is  $\zeta \in \text{dom}(t)$  with  $k_\delta(\zeta) = t(\zeta)$ . Set  $k = \bigcup_{\delta < \kappa} k_\delta$ .

Given  $f \in F$ , let  $\delta_f < \kappa$  be such that  $\gamma_{\delta_f} \leq \beta_f < \gamma_{\delta_f+1}$ . Then  $f \upharpoonright [\beta_f, \beta_f + \nu) \in W_{\delta_f}$ . Hence  $k(\zeta) = f(\zeta)$  for some  $\zeta \in [\beta_f, \beta_f + \nu)$ .  $\square$

**QUESTION.** Is it consistent that  $\kappa$  is a successor cardinal and  $\mathfrak{U}_\kappa < \mathfrak{d}_\kappa$ ?

**QUESTION.** Is it consistent that  $\kappa$  is a successor cardinal such that  $2^{<\kappa} = \kappa$  and  $\text{cov}(\mathbf{M}_{\kappa, \kappa}) < \mathfrak{U}_\kappa$ ?

Let us now consider the case when  $\kappa$  is a limit cardinal. By a result of Bartoszyński [B] and Miller [Mil1],  $\mathfrak{U}_\omega = \text{cov}(\mathbf{M}_{\omega, \omega})$ . Landver [L2] was able to show that this fact generalizes to uncountable inaccessible cardinals:

**THEOREM 4.7.** If  $\kappa$  is an inaccessible cardinal, then  $\mathfrak{U}_\kappa = \text{cov}(\mathbf{M}_{\kappa, \kappa})$ .

**QUESTION.** Is it consistent that  $\kappa$  is a limit cardinal and  $\text{cov}(\mathbf{M}_{\kappa, \kappa}) < \mathfrak{U}_\kappa$ ?

## 5. Weak selectivity

The following is due to Baumgartner, Taylor and Wagon [BauTW].

**PROPOSITION 5.1.** If  $\kappa$  is a successor cardinal, then every ideal on  $\kappa$  is a weak  $Q$ -point.

By Proposition 5.1 and Theorem 2.2  $\mathbf{non}_\kappa(\text{weakly selective}) = \mathfrak{d}_\kappa$  if  $\kappa$  is a successor cardinal. The remainder of the section is primarily concerned with the value of  $\mathbf{non}_\kappa(\text{weakly selective})$  in the case when  $\kappa$  is a limit cardinal.

**Remark.** It is easy to see that  $\kappa^+ \leq \mathbf{non}_\kappa(\text{weak } Q\text{-point})$  if  $\kappa$  is a limit cardinal.

**Definition.** An ideal  $J$  on  $\kappa$  is a weak semi- $Q$ -point if given  $A \in J^+$  and a  $< \kappa$ -to-one function  $f$  from  $A$  to  $\kappa$ , there is  $C \in J^+ \cap P(A)$  such that  $|C \cap f^{-1}(\{\alpha\})| \leq |\alpha|$  for every  $\alpha \in \kappa$ .  
 $J$  is weakly semiselective if  $J$  is both a weak semi- $Q$ -point and a weak  $P$ -point.  
 $J$  is weakly rapid if given  $A \in J^+$  and  $f \in {}^\kappa \kappa$ , there is  $C \in J^+ \cap P(A)$  such that  $\text{o.t.}(C \cap f(\alpha)) \leq \alpha + 1$  for every  $\alpha \in \kappa$ .

**Remark.** It is simple to see that every weak  $Q$ -point ideal on  $\kappa$  is weakly rapid, and every weakly rapid ideal on  $\kappa$  is a weak semi- $Q$ -point.

Every weak semi- $Q$ -point ideal on  $\omega$  is weakly rapid ([MP1]). We will show that this does not generalize.

**Definition.** An ideal  $J$  on  $\kappa$  is a semi- $Q$ -point if given a  $< \kappa$ -to-one function  $f$  from  $\kappa$  to  $\kappa$ , there is  $B \in J$  such that  $|f^{-1}(\{\alpha\}) - B| \leq |\alpha|$  for every  $\alpha \in \kappa$ .

**PROPOSITION 5.2.** Suppose  $\kappa$  is a limit cardinal. Then there exists a semi- $Q$ -point ideal on  $\kappa$  that is not weakly rapid.

**Proof.** Let  $Y$  be the set of all infinite cardinals  $< \kappa$ . Select  $h \in {}^Y \kappa$  so that (a)  $h(\mu)$  is a regular infinite cardinal  $\leq \mu$  for every  $\mu \in Y$ , and (b)  $\{\mu \in Y : h(\mu) \geq \theta\}$  is stationary in  $\kappa$  for every  $\theta \in Y$ . For  $A \subseteq \kappa$  and  $\theta \in Y$ , let  $T_\theta^A$  be the set of all  $\mu \in Y$  such that  $h(\mu) \geq \theta$  and  $|A \cap [\mu, \mu + h(\mu))| = h(\mu)$ . Now let  $J_h$  be the set of all  $A \subseteq \kappa$  such that  $T_\theta^A$  is a nonstationary subset of  $\kappa$  for some  $\theta \in Y$ . It is simple to check that  $J_h$  is an ideal on  $\kappa$ .

Let us remark in passing that if  $\kappa$  is weakly Mahlo and  $h$  is defined by  $h(\mu) = \omega$  if  $\mu$  is singular, and  $h(\mu) = \mu$  otherwise, then a subset  $A$  of  $\kappa$  lies in  $J_h$  if and only if the set of all  $\mu \in Y$  such that  $\mu$  is regular and  $|A \cap [\mu, \mu + \mu)| = \mu$  is nonstationary in  $\kappa$ .

Let us show that  $J_h$  is a semi- $Q$ -point. Thus fix a  $< \kappa$ -to-one function  $f : \kappa \rightarrow \kappa$ . Then

$$C = \left\{ \mu \in Y : \mu = \bigcup_{\alpha < \mu} f^{-1}(\{\alpha\}) \right\}$$

is a closed unbounded subset of  $\kappa$ . Set  $Q = \bigcup_{\mu \in C} [\mu, \mu + h(\mu))$ . It is immediate that  $\kappa - Q \in J_h$ . Now fix  $\alpha \in \kappa$  such that  $Q \cap f^{-1}(\{\alpha\}) \neq \emptyset$ . Pick  $\nu \in C$  so that

$$[\nu, \nu + h(\nu)) \cap f^{-1}(\{\alpha\}) \neq \emptyset.$$

Clearly,  $\alpha \geq \nu$  and  $\nu \in f^{-1}(\{\alpha\})$ . Let  $\rho$  be the least element of  $C$  that is  $> \nu$ . Then  $\alpha < \rho$  and  $f^{-1}(\{\alpha\}) \subseteq \rho$ . Thus

$$Q \cap f^{-1}(\{\alpha\}) \subseteq [\nu, \nu + h(\nu))$$

and consequently

$$|Q \cap f^{-1}(\{\alpha\})| \leq h(\nu) \leq \nu \leq |\alpha|.$$

It remains to show that  $J$  is not weakly rapid. Fix  $D \in J_h^+$ . Then

$$S = \{\mu \in T_\omega^D : |T_\omega^D \cap \mu| = \mu\}$$

is a stationary subset of  $\kappa$ . Given  $\mu \in S$ ,  $|D \cap \mu| = \mu$  since

$$D \cap [\rho, \rho + h(\rho)) \subset D \cap \mu$$

for every  $\rho \in \mu \cap T_\omega^D$ , and hence

$$\text{o.t.}(D \cap (\mu + h(\mu))) > \mu + 1.$$

□

**THEOREM 5.3.** Suppose  $\kappa$  is a limit cardinal. Then  $\mathfrak{U}_\kappa \leq \mathbf{non}_\kappa(\text{weak semi-}Q\text{-point})$ .



**Proof.** Let  $J$  be an ideal on  $\kappa$  with  $\text{cof}(J) < \mathfrak{U}_\kappa$ . Let us show that  $J$  is a weak semi- $Q$ -point. Thus fix  $A \in J^+$  and a  $< \kappa$ -to-one function  $f : A \rightarrow \kappa$ . Select  $B_\beta \in J$  for  $\beta < \text{cof}(J)$  so that  $J = \bigcup_{\beta < \text{cof}(J)} P(B_\beta)$ .

For  $\beta < \text{cof}(J)$ , define  $g_\beta \in {}^\kappa \kappa$  by :

$$g_\beta(\alpha) = \text{least element of } \left( \bigcup_{\gamma > \alpha} f^{-1}(\{\gamma\}) \right) - B_\beta.$$

There is  $h \in {}^\kappa \kappa$  such that  $\{\alpha \in \kappa : g_\beta(\alpha) = h(\alpha)\} \neq \emptyset$  for every  $\beta < \text{cof}(J)$ . Define  $C \subseteq \text{ran}(h)$  by :  $h(\alpha) \in C$  just in case  $h(\alpha) \in \bigcup_{\gamma > \alpha} f^{-1}(\{\gamma\})$ . Then clearly  $C \in J^+ \cap P(A)$ . Moreover,  $C \cap f^{-1}(\{\alpha\}) \subseteq \{h(\gamma) : \gamma < \alpha\}$  for every  $\alpha < \kappa$ .  $\square$

**THEOREM 5.4.** Suppose that  $\kappa$  is a limit cardinal and  $2^{<\kappa} = \kappa$ . Then

$$\overline{\mathbf{non}}_\kappa(\text{weakly semiselective}) \leq \mathfrak{U}_\kappa \leq \mathbf{non}_\kappa(\text{weak } Q\text{-point}).$$

**Proof.** The proof of the first inequality is an easy modification of that of Lemma 6.1 in [MP1] (which should be corrected by substituting “ $e \in [\omega]^{<\omega}$  such that  $B \subseteq \bigcup_{j \in e} \omega^{E_j} \cup \bigcup_{f \in z} B_f$ ” for “ $e \in [\bigcup_{j \in \omega} \omega^{E_j}]^{<\omega}$  such that  $B \subseteq e \cup \bigcup_{f \in z} B_f$ ”). The second inequality is proved as Proposition 5.3 in [MP1].  $\square$

**Remark.** Suppose that  $\kappa$  is a limit cardinal,  $2^{<\kappa} = \kappa$  and  $\mathbf{non}_\kappa(\text{weakly semiselective}) < \kappa^{+\omega}$ . Then by Proposition 4.1 and Theorems 2.2 and 5.4,

$$\mathfrak{U}_\kappa = \mathbf{non}_\kappa(\text{weakly selective}) = \mathbf{non}_\kappa(\text{weakly semiselective}).$$

**Remark.** It is consistent (see [MP1]) that  $\mathfrak{U}_\omega < \mathbf{non}_\omega(\text{weak } Q\text{-point})$ , and that  $\mathbf{non}_\omega(\text{weak } Q\text{-point}) < \mathbf{non}_\omega(\text{weak semi-}Q\text{-point})$ . We do not know whether these results can be generalized.

**QUESTION.** Is it consistent that  $\kappa$  is a limit cardinal,  $2^{<\kappa} > \kappa$  and  $\kappa^+ < \mathbf{non}_\kappa(\text{weak } Q\text{-point})$  ?

**QUESTION.** By a result of [MP1],  $\text{cf}(\mathbf{non}_\omega(\text{weak } Q\text{-point})) > \omega$ . Does this generalize ?

## 6. $\mathbf{non}_\kappa(J^+ \rightarrow (J^+, \theta)^2)$

In this section we use standard material to discuss the value of  $\mathbf{non}_\kappa(J^+ \rightarrow (J^+, \theta)^2)$  for a cardinal  $\theta \in [3, \kappa]$ .

**THEOREM 6.1.**

- (i)  $\mathfrak{d}_\kappa \geq \mathbf{non}_\kappa(J^+ \rightarrow (J^+, 3)^2)$ .
- (ii)  $\overline{\mathfrak{d}}_\kappa \geq \overline{\mathbf{non}}_\kappa(J^+ \rightarrow (J^+, 3)^2)$ .
- (iii)  $\overline{\mathbf{non}}_\kappa(\text{weak } P\text{-point}) \geq \overline{\mathbf{non}}_\kappa(J^+ \rightarrow (J^+, \omega)^2)$ .

**Proof.** (i) and (ii) : By a straightforward generalization of Lemma 4.4 in [M2], there exists an ideal  $J$  on  $\kappa$  such that  $\overline{\text{cof}}(J) \leq \overline{\mathfrak{d}}_\kappa$ ,  $\text{cof}(J) \leq \mathfrak{d}_\kappa$  and  $J^+ \not\rightarrow (J^+, 3)^2$ .

(iii) : Baumgartner, Taylor and Wagon [BauTW] established that if  $J$  is an ideal on  $\kappa$  such that  $J^+ \rightarrow (J^+, \omega)^2$ , then  $J$  is a weak  $P$ -point.  $\square$

**Definition.** Given an ideal  $J$  on  $\kappa$ ,  $A \in J^+$  and  $F : \kappa \times \kappa \rightarrow 2$ ,  $(J, A, F)$  is 0-good if there is  $D \in J^+ \cap P(A)$  such that  $\{\beta \in D : F(\alpha, \beta) = 1\} \in J$  for every  $\alpha \in D$ .

The following is readily checked.

**LEMMA 6.2.** Suppose that  $J$  is weakly selective and  $(J, A, F)$  is 0-good, where  $J$  is an ideal on  $\kappa$ ,  $A \in J^+$  and  $F : \kappa \times \kappa \rightarrow 2$ . Then there is  $B \in J^+ \cap P(A)$  such that  $F$  is constantly 0 on  $[B]^2$ .

**LEMMA 6.3.** Suppose that  $(J, A, F)$  is not 0-good, where  $J$  is an ideal on  $\kappa$ ,  $A \in J^+$  and  $F : \kappa \times \kappa \rightarrow 2$ . Then :

- (i) There is  $B \subseteq A$  such that  $\text{o.t.}(B) = \omega + 1$  and  $F$  is identically 1 on  $[B]^2$ .
- (ii) Suppose that  $\mathfrak{a}_J > \kappa$  and  $\theta$  is an uncountable cardinal  $< \kappa$  such that  $\kappa \rightarrow (\kappa, \theta)^2$ . Then there is  $C \subseteq A$  such that  $\text{o.t.}(C) = \theta + 1$  and  $F$  is identically 1 on  $[C]^2$ .

**Proof.** The proof is similar to that of Lemma 10.4 below. □

**THEOREM 6.4.**

- (i)  $\overline{\mathbf{non}}_\kappa(J^+ \rightarrow (J^+, \omega + 1)^2) \geq \overline{\mathbf{non}}_\kappa(\text{weakly selective})$ .
- (ii) Suppose that  $\theta$  is an infinite cardinal  $< \kappa$  such that  $\kappa \rightarrow (\kappa, \theta)^2$ . Then

$$\mathbf{non}_\kappa(J^+ \rightarrow (J^+, \theta + 1)^2) \geq \mathbf{non}_\kappa(\text{weakly selective}).$$

**Proof.** (i) : Baumgartner, Taylor and Wagon [BauTW] showed that  $J^+ \rightarrow (J^+, \omega + 1)^2$  for every weakly selective ideal  $J$  on  $\kappa$ .

(ii) : By Lemmas 6.2 and 6.3. □

**Remark.** Suppose that  $\kappa$  is a successor cardinal and  $\theta$  is cardinal  $\geq 2$  such that  $\kappa \rightarrow (\kappa, \theta)^2$ . Then by Theorems 6.1 (i), 6.4 (ii) and 2.2 and Proposition 5.1,  $\mathfrak{d}_\kappa = \mathbf{non}_\kappa(J^+ \rightarrow (J^+, \theta + 1)^2)$ .

**Remark.** It is consistent (see [M2]) that  $\mathfrak{d} > \mathbf{non}_\omega(J^+ \rightarrow (J^+, 3)^2)$ . We do not know whether this can be generalized.

**THEOREM 6.5.** Suppose  $\kappa$  is a weakly compact cardinal. Then :

- (i)  $\overline{\mathbf{non}}_\kappa(\text{weak } Q\text{-point}) \geq \overline{\mathbf{non}}_\kappa(J^+ \rightarrow (J^+, \kappa)^2)$ .
- (ii)  $\mathbf{non}_\kappa(J^+ \rightarrow (J^+)^2) = \mathbf{non}_\kappa(J^+ \rightarrow (J^+, \kappa)^2) = \mathbf{non}_\kappa(\text{weakly selective})$ .

**Proof.** The result follows from Theorems 2.2 and 6.1 (i) and the following two well-known facts : (1) Every ideal  $J$  on  $\kappa$  such that  $J^+ \rightarrow (J^+, \kappa)^2$  is a weak  $Q$ -point ; (2) If  $\kappa$  is weakly compact, then  $J^+ \rightarrow (J^+)^2$  for every weakly selective ideal  $J$  on  $\kappa$  such that  $\mathfrak{a}_J > \kappa$ . □

## 7. $\mathbf{non}_\kappa(J^+ \longrightarrow [J^+]_\rho^2)$

In this section we consider the cardinal  $\mathbf{non}_\kappa(J^+ \longrightarrow [J^+]_\rho^2)$ , where  $3 \leq \rho \leq \kappa$ , about which little is known. We begin with the case where  $\rho = 3$ . The following is due to Blass [Bl].

**LEMMA 7.1.** *Suppose  $J$  is an ideal on  $\kappa$  such that  $J^+ \longrightarrow [J^+]_3^2$ . Then  $J$  is a weak  $P$ -point.*

**Proof.** Fix  $A \in J^+$  and  $f \in {}^A\kappa$  with  $\{f^{-1}(\{\gamma\}) : \gamma \in \kappa\} \subseteq J$ . Define  $g : [A]^2 \longrightarrow 3$  by stipulating that  $g(\alpha, \beta) = 0$  if and only if  $f(\alpha) < f(\beta)$ , and  $g(\alpha, \beta) = 1$  if and only if  $f(\alpha) = f(\beta)$ . There are  $B \in J^+ \cap P(A)$  and  $i < 3$  such that  $i \notin g''[B]^2$ . It is simple to see that  $i \neq 0$ , so  $f$  is  $< \kappa$ -to-one on  $B$ .  $\square$

The following is proved by adapting an argument of Baumgartner and Taylor [BauT].

**LEMMA 7.2.** *Suppose  $J$  is an ideal on  $\kappa$  such that  $J^+ \longrightarrow [J^+]_3^2$ , and  $(J, A, F)$  is 0-good, where  $A \in J^+$  and  $F : \kappa \times \kappa \longrightarrow 2$ . Then either there exists  $C \in J^+ \cap P(A)$  such that  $F$  is constantly 0 on  $[C]^2$ , or for every  $\delta < \kappa$ , there exists  $Q \subseteq A$  such that  $\text{o.t.}(Q) = \delta$  and  $F$  is constantly 1 on  $[Q]^2$ .*

**Proof.** Select  $B \in J^+ \cap P(A)$  so that  $\{\beta \in B : F(\alpha, \beta) = 1\} \in J$  for every  $\alpha \in B$ . By Lemma 7.1, there exists  $S \in J^+ \cap P(B)$  so that  $|\{\beta \in S : F(\alpha, \beta) = 1\}| < \kappa$  for every  $\alpha \in S$ . Define  $\delta_\xi$  for  $\xi < \kappa$  by :

- (i)  $\delta_0 = \cap S$ ;
- (ii)  $\delta_{\xi+1}$  = the least  $\zeta < \kappa$  with the property that  $\zeta > \beta$  for every  $\beta \in S$  such that  $F(\alpha, \beta) = 1$  for some  $\alpha \in S \cap \delta_\zeta$  ;
- (iii)  $\delta_\xi = \bigcup_{\zeta < \xi} \delta_\zeta$  if  $\xi$  is a limit ordinal  $> 0$ .

Let  $X$  be the set of all limit ordinals  $< \kappa$ . For  $\eta \in X$ ,  $n \in \omega$  and  $j < 2$ , set

$$d_{\eta,n}^j = S \cap [\delta_{\eta+2n+j}, \delta_{\eta+2n+j+1}).$$

For  $j < 2$ , let

$$D^j = \cup \{d_{\eta,n}^j : \eta \in X \text{ and } n \in \omega\}.$$

Select  $k < 2$  so that  $D^k \in J^+$ . Notice that  $F(\alpha, \beta) = 0$  if  $(\alpha, \beta) \in [D^k]^2$  and  $\{\alpha, \beta\} \not\subseteq d_{\eta,n}^k$  for all  $\eta \in X$  and  $n \in \omega$ .

Define  $h : [D^k]^2 \longrightarrow 3$  by stipulating that  $h(\alpha, \beta) = 0$  if and only if  $\{\alpha, \beta\} \not\subseteq d_{\eta,n}^k$  for all  $\eta \in X$  and  $n \in \omega$ , and  $h(\alpha, \beta) = 1$  if and only if  $F(\alpha, \beta) = 1$ . There are  $W \in J^+ \cap P(D^k)$  and  $i < 3$  so that  $i \notin h''[W]^2$ . Clearly,  $i \neq 0$ . If  $i = 1$ ,  $F$  is identically 0 on  $[W]^2$ . Now assume  $i = 2$ . Let  $Z$  be the set of all  $(\eta, n) \in X \times \omega$  such that  $W \cap d_{\eta,n}^k \neq \emptyset$ . Suppose that there is  $\gamma < \kappa$  such that  $\text{o.t.}(W \cap d_{\eta,n}^k) \leq \gamma$  for every  $(\eta, n) \in Z$ . Then there exists  $C \in J^+ \cap P(W)$  such that  $|C \cap d_{\eta,n}^k| = 1$  for any  $(\eta, n) \in Z$ . Clearly,  $F$  takes the constant value 0 on  $[T]^2$ .  $\square$

**PROPOSITION 7.3.** *Suppose  $\theta \in (2, \kappa)$  is a cardinal such that  $\kappa \longrightarrow (\kappa, \theta)^2$ . Then*

$$\mathbf{non}_\kappa(J^+ \longrightarrow [J^+]_3^2) \leq \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \theta + 1)^2).$$

**Proof.** By Theorem 2.2 and Lemmas 6.3, 7.1 and 7.2.  $\square$

Let us now consider the partition relation  $J^+ \longrightarrow [J^+]_\kappa^2$ . We begin with the following observation.

**PROPOSITION 7.4.** Suppose  $\kappa$  is inaccessible. Then there is an ideal  $J$  on  $\kappa$  such that (a)  $J^+ \not\rightarrow [J^+]^2_\kappa$ , (b)  $J$  is not a weak semi- $Q$ -point, (c)  $\mathfrak{a}_J > \kappa$ , and (d)  $J^+ \rightarrow (J^+, \alpha)^2$  for every  $\alpha < \kappa$ .

**Proof.** Let  $\langle \rho_\alpha : \alpha < \kappa \rangle$  be the increasing enumeration of all strong limit infinite cardinals  $< \kappa$ . Let  $Z$  be the set of all regular infinite cardinals  $< \kappa$ . For  $\mu \in Z$ , set  $\nu_\mu = (\rho_\mu)^{++}$ . Then  $\nu_\mu \not\rightarrow [\nu_\mu]_{\nu_\mu}^2$  by a result of Todorćević [To1]. On the other hand, by a result of Erdős and Rado (see [EHMáR], Corollary 17.5),  $\nu_\mu \rightarrow (\nu_\mu, \tau)^2$  for every infinite cardinal  $\tau < \mu$ . Pick pairwise disjoint  $A_\mu$  for  $\mu \in Z$  so that  $|A_\mu| = \nu_\mu$  for any  $\mu \in Z$ , and  $\bigcup_{\mu \in Z} A_\mu = \kappa$ . Let  $J$  be the set of all  $B \subseteq \kappa$  such that

$$|\{\mu \in Z : |B \cap A_\mu| = \nu_\mu\}| < \kappa.$$

It is simple to see that  $J$  is an ideal on  $\kappa$ .

For  $\mu \in Z$ , pick  $g_\mu : [A_\mu]^2 \rightarrow \nu_\mu$  so that  $g_\mu''[B]^2 = \nu_\mu$  for every  $B \subseteq A_\mu$  with  $|B| = \nu_\mu$ . Let  $G : [\kappa]^2 \rightarrow \kappa$  be such that  $\bigcup_{\mu \in Z} g_\mu \subseteq G$ . Then clearly  $G''[C]^2 = \kappa$  for any  $C \in J^+$ .

Define  $f \in {}^\kappa \kappa$  by stipulating that  $f^{-1}(\{\mu\}) = A_\mu$  for every  $\mu \in Z$ . Clearly, there is no  $S \in J^+$  so that  $|S \cap f^{-1}(\{\alpha\})| \leq |\alpha|$  for all  $\alpha < \kappa$ . Hence  $J$  is not a weak semi- $Q$ -point.

Let us next show that  $\mathfrak{a}_J > \kappa$ . Thus suppose that  $B_\alpha \in J^+$  for  $\alpha < \kappa$ , and  $B_\alpha \cap B_\beta \in J$  whenever  $\beta < \alpha < \kappa$ . Select a strictly increasing function  $k : \kappa \rightarrow Z$  so that

$$|(B_\alpha - (\bigcup_{\beta < \alpha} B_\beta)) \cap A_{k(\alpha)}| = \nu_{k(\alpha)}$$

for any  $\alpha < \kappa$ . Set

$$T = \bigcup_{\alpha < \kappa} ((B_\alpha - (\bigcup_{\beta < \alpha} B_\beta)) \cap A_{k(\alpha)}).$$

Then  $T \in J^+$  and moreover  $|T \cap B_\alpha| < \kappa$  for every  $\alpha < \kappa$ .

It remains to prove (d). Thus fix  $A \in J^+$  and  $F : \kappa \times \kappa \rightarrow 2$ . Suppose that there is  $\eta < \kappa$  such that for every  $Q \subseteq A$  with  $\text{o.t.}(Q) = \eta$ ,  $F$  is not constantly 1 on  $[Q]^2$ . Since by Theorem 17.1 of [EHMáR]  $\kappa \rightarrow (\kappa, \alpha)^2$  for every  $\alpha < \kappa$ , it follows from Lemma 6.3 that  $(J, A, F)$  is 0-good. Select  $D \in J^+ \cap P(A)$  so that  $\{\beta \in D : F(\alpha, \beta) = 1\} \in J$  for every  $\alpha \in D$ . Define  $D_\gamma$  for  $\gamma < \kappa$  and a strictly increasing function  $h : \kappa \rightarrow Z$  so that

$$(0) \quad D_\gamma = D - \left( \bigcup_{\delta < \gamma} \bigcup_{\alpha \in D_\delta \cap A_{h(\delta)}} \{\beta \in D : F(\alpha, \beta) = 1\} \right);$$

$$(1) \quad |D_\gamma \cap A_{h(\gamma)}| = \nu_{h(\gamma)}.$$

For  $\gamma \in (|\eta|^+, \kappa)$ , select  $X_\gamma \subseteq D_\gamma \cap A_{h(\gamma)}$  so that  $|X_\gamma| = \nu_{h(\gamma)}$  and  $F$  is constantly 0 on  $[X_\gamma]^2$ . Set  $Y = \bigcup_{|\eta|^+ < \gamma < \kappa} X_\gamma$ . Then clearly  $Y \in J^+ \cap P(A)$ . Moreover,  $F$  takes the constant value 0 on  $[Y]^2$ .  $\square$

**Remark.**  $J^+ \rightarrow (J^+, \kappa)^2$  does not necessarily imply that  $J^+ \rightarrow [J^+]^2_\kappa$ . This follows from the following two facts : (0) If  $\kappa$  is weakly compact, then there exists a normal ideal  $J$  on  $\kappa$  such that  $J^+ \rightarrow (J^+, \kappa)^2$  ([Bau1], [Bau2]) ; (1) Assuming  $V = L$ ,  $\kappa$  is completely ineffable if and only if there is a normal ideal  $J$  on  $\kappa$  such that  $J^+ \rightarrow [J^+]^2_\kappa$  ([M4]).

Recall that for  $S \subseteq \kappa$ ,  $\diamond_\kappa^*(S)$  means that there are  $s_\alpha \in P_{|\alpha|^+}(\alpha)$  for  $\alpha \in S$  such that for every  $A \subseteq \kappa$ , there exists a closed unbounded subset  $C$  of  $\kappa$  with the property that  $A \cap \alpha \in s_\alpha$  for every  $\alpha \in C \cap S$ .

**PROPOSITION 7.5.-** Suppose that  $\diamond_\kappa^*(S)$  holds for some stationary subset  $S$  of  $\kappa$ . Then  $\mathfrak{d}_\kappa \geq \mathfrak{non}_\kappa(J^+ \rightarrow [J^+]^2_\kappa)$  and  $\overline{\mathfrak{d}}_\kappa \geq \overline{\mathfrak{non}}_\kappa(J^+ \rightarrow [J^+]^2_\kappa)$ .

**Proof.** By a result of [M4], the hypothesis implies that  $NS_\kappa^+ \not\rightarrow [NS_\kappa^+]^2_\kappa$ .  $\square$

**Remark.** It is shown in [S] that if (a)  $\kappa$  is a successor cardinal  $\geq \omega_2$  with  $2^{<\kappa} = \kappa$ , and (b) setting  $\kappa = \nu^+$ ,  $\mu^\tau \leq \nu$  for every infinite cardinal  $\mu < \nu$ , where  $\tau = \aleph_1$  if  $\text{cf}(\nu) = \omega$  and  $\tau = \aleph_0$  otherwise, then there is a stationary subset  $S$  of  $\kappa$  such that  $\diamond_\kappa^*(S)$  holds.

**Remark.** We do not know whether it is consistent that the conclusion of Proposition 7.5 fails. Results of Section 15 (below) imply that

$$\mathbf{non}_\kappa(J^+ \rightarrow [J^+]_\kappa^2) \leq (\overline{\mathfrak{d}}_\kappa)^{<\kappa}$$

if  $\kappa$  is a limit cardinal such that  $2^{<\kappa} = \kappa$ .

## 8. $\mathfrak{d}_{\kappa,\lambda}^\kappa$

We now start our study of combinatorial properties of ideals on  $P_\kappa(\lambda)$ . The aim of this section is to present a two-cardinal version of Theorem 2.2.

**Definition.**  $\mathfrak{d}_{\kappa,\lambda}^\kappa$  is the least cardinality of any  $F \subseteq {}^\kappa(P_\kappa(\lambda))$  with the property that for every  $g \in {}^\kappa(P_\kappa(\lambda))$ , there is  $f \in F$  such that  $g(\alpha) \subseteq f(\alpha)$  for all  $\alpha \in \kappa$ .

**Remark.** It is shown in [MP  S1] that  $\mathfrak{d}_{\kappa,\lambda}^\kappa = \mathfrak{d}_\kappa \cdot u(\kappa^+, \lambda)$ .

**Definition.** Given an ideal  $H$  on  $P_\kappa(\lambda)$ ,  $\mathcal{M}_H^{\geq \kappa}$  is the set of all  $Q \subseteq H^+$  such that (i)  $|Q| \geq \kappa$ , (ii)  $A \cap B \in H$  for all  $A, B \in Q$  with  $A \neq B$ , and (iii) for every  $C \in H^+$ , there is  $A \in Q$  with  $A \cap C \in H^+$ .  $\mathfrak{a}_H$  is the least cardinality of any member of  $\mathcal{M}_H^{\geq \kappa}$  if  $\mathcal{M}_H^{\geq \kappa} \neq \emptyset$ , and  $2^{(\lambda^{<\kappa})^+}$  otherwise.

The following is proved as Proposition 11.2 of [MP2].

**PROPOSITION 8.1.** *Given a  $\kappa$ -normal ideal  $H$  on  $P_\kappa(\lambda)$ , the following are equivalent :*

- (i)  $\mathfrak{a}_H = \kappa$ .
- (ii)  $\text{sat}(H) > \kappa$ .

**COROLLARY 8.2.** *Let  $A \in (NS_{\kappa,\lambda}^\kappa)^+$  and set  $H = NS_{\kappa,\lambda}^\kappa \restriction A$ . Then  $\mathfrak{a}_H = \kappa$ .*

**Proof.** The result follows from Proposition 8.1 since  $\text{sat}(H) > \kappa$  by a result of Abe [A].  $\square$

The following is proved as Proposition 11.1 (ii) of [MP2].

**PROPOSITION 8.3.** *Given an ideal  $H$  on  $P_\kappa(\lambda)$ , the following are equivalent :*

- (i)  $\mathfrak{a}_H = \kappa$ .
- (ii) *There exist  $A_\alpha \in H^+$  for  $\alpha < \kappa$  such that (a)  $A_\alpha \subseteq A_\beta$  whenever  $\beta < \alpha < \kappa$ , and (b) for every  $C \in H^+$ , there is  $\alpha < \kappa$  such that  $C - A_\alpha \in H^+$ .*

**Definition.** An ideal  $H$  on  $P_\kappa(\lambda)$  is a weak  $\pi$ -point if given  $f \in {}^\kappa H$  and  $A \in H^+$ , there is  $B \in H^+ \cap P(A)$  such that  $B \cap f(\alpha) \in I_{\kappa,\lambda}$  for every  $\alpha \in \kappa$ .

**THEOREM 8.4.** *Let  $H$  be an ideal on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathfrak{d}_{\kappa,\lambda}^\kappa$ . Then  $\mathfrak{a}_H > \kappa$  and  $H$  is a weak  $\pi$ -point.*

**Proof.** Let  $A_\alpha \in H^+$  for  $\alpha < \kappa$  be such that  $A_\alpha \subseteq A_\beta$  for all  $\beta < \alpha$ . Select  $X \subseteq H$  so that  $|X| = \text{cof}(H)$  and  $H = \bigcup_{B \in X} P(B)$ . For  $B \in X$ , define  $f_B \in {}^\kappa(P_\kappa(\lambda))$  so that  $f_B(\alpha) \in A_\alpha - B$ . There is  $g \in {}^\kappa(P_\kappa(\lambda))$  such

that  $\{\alpha < \kappa : g(\alpha) \not\subseteq f_B(\alpha)\} \neq \emptyset$  for every  $B \in X$ . Set  $C = \bigcup_{\alpha \in \kappa} \{a \in A_\alpha : g(\alpha) \not\subseteq a\}$ . Then  $C \in H^+$ , and moreover  $C - A_\alpha \in I_{\kappa, \lambda}$  for any  $\alpha < \kappa$ .

**Definition.**  $\overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$  is the least cardinality of any  $X \subseteq {}^\kappa(P_\kappa(\lambda))$  with the property that for every  $g \in {}^\kappa(P_\kappa(\lambda))$ , there is  $x \in P_\kappa(X)$  such that  $g(\alpha) \subseteq \bigcup_{f \in x} f(\alpha)$  for every  $\alpha < \kappa$ .

**Remark.** It is shown in [MRoS] that  $\overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa = \overline{\mathfrak{d}}_\kappa \cdot \text{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$ , where  $\text{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$  denotes the least cardinality of any  $X \subseteq P_{\kappa^+}(\lambda)$  such that for every  $b \in P_{\kappa^+}(\lambda)$ , there is  $x \in P_\kappa(X)$  with  $b \subseteq \bigcup x$ .

**Remark.** It is immediate that  $I_{\kappa, \lambda}$  is a weak  $\pi$ -point. On the other hand  $\mathfrak{a}_{I_{\kappa, \lambda}} > \kappa$  does not necessarily hold. In fact if  $\text{cf}(\lambda) \neq \kappa$  and  $\overline{\mathfrak{d}}_{\kappa, \sigma}^\kappa \leq \lambda$  for every cardinal  $\sigma \in [\kappa, \lambda)$ , then  $\mathfrak{a}_{I_{\kappa, \lambda}} = \kappa$  ([M6]).

**LEMMA 8.5.** Suppose that  $H$  is an ideal on  $P_\kappa(\lambda)$  with  $\mathfrak{a}_H = \kappa$ . Then there is an ideal  $K$  on  $P_\kappa(\lambda)$  such that (a)  $K$  is not a weak  $\pi$ -point, (b)  $\text{cof}(K) \leq \text{cof}(H)$ , and (c)  $\overline{\text{cof}}(K) \leq \overline{\text{cof}}(H)$ .

**Proof.** Select  $A_\alpha \in H^+$  for  $\alpha < \kappa$  so that  $(\alpha) A_\alpha \subseteq A_\beta$  whenever  $\beta < \alpha < \kappa$ , and  $(\beta)$  for any  $C \in H^+$ , there is  $\alpha < \kappa$  with  $C - A_\alpha \in H^+$ . Let  $K$  be the set of all  $B \subseteq P_\kappa(\lambda)$  such that  $B \cap A_\alpha \in H$  for some  $\alpha < \kappa$ . It is simple to check that  $K$  is as desired.  $\square$

## THEOREM 8.6.

- (i) There is an ideal  $H$  on  $P_\kappa(\lambda)$  such that (a)  $\mathfrak{a}_H = \kappa$ , (b)  $\text{cof}(H) = \mathfrak{d}_{\kappa, \lambda}^\kappa$ , and (c)  $\overline{\text{cof}}(H) \leq \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$ .
- (ii) There is an ideal  $K$  on  $P_\kappa(\lambda)$  such that (a)  $K$  is not a weak  $\pi$ -point, (b)  $\text{cof}(K) = \mathfrak{d}_{\kappa, \lambda}^\kappa$ , and (c)  $\overline{\text{cof}}(K) \leq \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$ .

**Proof.** (i) : Set  $H = NS_{\kappa, \lambda}^\kappa$ . Then  $\mathfrak{a}_H = \kappa$  by Corollary 8.2. Moreover,  $\text{cof}(H) = \mathfrak{d}_{\kappa, \lambda}^\kappa$  ([MPéS1]) and  $\overline{\text{cof}}(H) = \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$  ([MRoS]).

(ii) : By (i), Lemma 8.5 and Theorem 8.4.  $\square$

**Remark.** Theorem 8.6 is not optimal, even under GCH. In fact, suppose that the GCH holds,  $\lambda = \sigma^+$ , where  $\sigma$  is a cardinal of cofinality  $< \kappa$ , and  $\kappa$  is not the successor of a cardinal of cofinality  $\leq \text{cf}(\sigma)$ . Then  $\overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa = \lambda$  ([MRoS]). Moreover, there is  $A \in (NS_{\kappa, \lambda}^\kappa)^+$  such that  $\overline{\text{cof}}(NS_{\kappa, \lambda}^\kappa \restriction A) = \sigma$  ([MPéS2]). Hence there is by Corollary 8.2 an ideal  $H$  on  $P_\kappa(\lambda)$  (namely  $H = NS_{\kappa, \lambda}^\kappa \restriction A$ ) such that  $\overline{\text{cof}}(H) < \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$  and  $\mathfrak{a}_H = \kappa$ , and by Lemma 8.5 an ideal  $K$  on  $P_\kappa(\lambda)$  such that  $\overline{\text{cof}}(K) < \overline{\mathfrak{d}}_{\kappa, \lambda}^\kappa$  and  $K$  is not a weak  $\pi$ -point.

## 9. Weak $\chi$ -pointness

**Definition.** An ideal  $H$  on  $P_\kappa(\lambda)$  is a weak  $\chi$ -point if given  $A \in H^+$  and  $g \in {}^\kappa(P_\kappa(\lambda))$ , there is  $B \in H^+ \cap P(A)$  such that  $g(\cup(a \cap \kappa)) \subseteq b$  for all  $a, b \in B$  with  $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ .

Our primary concern in this section is with the problem of determining when  $I_{\kappa, \lambda}$  is a weak  $\chi$ -point. We will first give a sufficient condition and then prove that this condition is necessary if  $\kappa$  is inaccessible.

The following is proved as Lemma 2.1 in [M2].

**THEOREM 9.1.** Let  $H$  be an ideal on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \text{cov}(\mathbf{M}_{\kappa, \kappa})$ . Then  $H$  is a weak  $\chi$ -point.

**QUESTION.** Is it consistent that  $2^{<\kappa} > \kappa$  and  $I_{\kappa, \kappa^+}$  is a weak  $\chi$ -point?

**THEOREM 9.2.** Suppose that for all  $A \in I_{\kappa, \lambda}^+$  with  $A \subseteq \{a : \cup(a \cap \kappa) \in a\}$ , there is  $B \in I_{\kappa, \lambda}^+ \cap P(A)$  such that  $\cup(a \cap \kappa) \in b$  for all  $a, b \in B$  with  $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ . Then  $\sigma < \mathbf{non}_\kappa$  (weakly selective) for every  $\sigma \in \mathcal{K}(\kappa, \lambda)$ .

**Proof.** Suppose that  $T \subseteq P_\kappa(\lambda - \kappa)$  is such that  $|T \cap P(a)| < \kappa$  for every  $a \in P_\kappa(\lambda)$ , and  $J$  is an ideal on  $\kappa$  with  $\overline{cof}(J) \leq |T|$ . Select  $D_d \in J$  for  $d \in T$  so that for every  $W \in J$ , there is  $u \in P_\kappa(T) - \{\phi\}$  with  $W \subseteq \bigcup_{d \in u} D_d$ . Now fix  $G_\alpha \in J$  for  $\alpha < \kappa$ . Define  $A \subseteq P_\kappa(\lambda)$  by stipulating that  $a \in A$  if and only if there is  $\delta < \kappa$  such that (a)  $\delta = \max(a \cap \kappa)$ , (b)  $\delta \notin \bigcup_{d \in T \cap P(a)} D_d$ , and (c)  $\delta \notin G_\alpha$  for every  $\alpha \in a \cap \delta$ .

Let us show that  $A \in I_{\kappa, \lambda}^+$ . Given  $c \in P_\kappa(\lambda)$ , pick  $\delta < \kappa$  so that  $\delta \notin \bigcup_{d \in T \cap P(c)} D_d$  and for every  $\alpha \in c \cap \kappa$ ,  $\delta > \alpha$  and  $\delta \notin G_\alpha$ . Set  $e = c \cup \{\delta\}$ . Then  $e \in A$ .

By our assumption there is  $B \in I_{\kappa, \lambda}^+ \cap P(A)$  such that  $\cup(a \cap \kappa) \in b$  for all  $a, b \in B$  with  $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ . Set  $C = \{\cup(a \cap \kappa) : a \in B\}$ . Then  $C \in J^+$ . Moreover,  $\xi \notin G_\zeta$  for all  $\zeta, \xi \in C$  with  $\zeta < \xi$ .  $\square$

We mention the following partial converse to Theorem 9.2.

**PROPOSITION 9.3.** Suppose that  $2^{<\kappa} = \kappa$  and  $H$  is an ideal on  $P_\kappa(\lambda)$  such that  $cof(H) < \mathbf{non}_\kappa$  (weakly selective). Then for all  $f \in {}^\kappa \kappa$  and  $A \in H^+$ , there is  $B \in H^+ \cap P(A)$  such that  $f(\cup(a \cap \kappa)) \subseteq b$  for all  $a, b \in B$  with  $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ .

**Proof.** Fix  $f \in {}^\kappa \kappa$  and  $A \in H^+$ . For  $D \subseteq P_\kappa(\kappa)$ , set  $Z_D = \{a \in P_\kappa(\lambda) : a \cap \kappa \in D\}$ . It is simple to see that (a)  $Z_{P_\kappa(\kappa)} = P_\kappa(\lambda)$ , (b)  $Z_{\bigcup \mathfrak{D}} Z_D$  for  $\mathfrak{D} \subseteq P(P_\kappa(\kappa))$ , (c)  $Z_D \in I_{\kappa, \lambda}$  for every  $D \subseteq P_\kappa(\kappa)$  with  $|D| = 1$ , and (d)  $Z_{D'} \subseteq Z_D$  for all  $D, D' \subseteq P_\kappa(\kappa)$  such that  $D' \subseteq D$ . Hence

$$K = \{D \subseteq P_\kappa(\kappa) : Z_D \in H \mid A\}$$

is a  $\kappa$ -complete ideal on  $P_\kappa(\kappa)$ . For  $C \subseteq P_\kappa(\lambda)$ , let  $W_C$  be the set of all  $d \in P_\kappa(\kappa)$  such that

$$\{a \in P_\kappa(\lambda) : a \cap \kappa = d\} \subseteq C.$$

If  $C \in H \mid A$ , then  $W_C \in K$  since  $Z_{W_C} \subseteq C$ . Moreover, if  $D \subseteq P_\kappa(\kappa)$  and  $Z_D \subseteq C \subseteq P_\kappa(\lambda)$ , then  $D \subseteq W_C$ . Hence

$$cof(K) \leq cof(H \mid A) \leq cof(H).$$

For  $d \in P_\kappa(\kappa)$ , let  $S_d$  be the set of all  $e \in P_\kappa(\kappa)$  such that  $f(\cup d) \not\subseteq e$  or  $\cup e \leq \cup d$ . Then  $S_d \in K$  since

$$\{a \in Z_{S_d} : f(\cup d) \cup \{(\cup d) + 1\} \subseteq a\} = \phi.$$

Select a bijection  $\ell : P_\kappa(\kappa) \rightarrow \kappa$ . Since  $cof(K) < \mathbf{non}_\kappa$  (weakly selective), there is  $D \in K^+$  such that  $e \notin S_d$  for all  $d, e \in D$  such that  $\ell(d) < \ell(e)$ . Set

$$B = A \cap Z_D = \{a \in A : a \cap \kappa \in D\}.$$

Then  $B \in H^+$ . Now fix  $a, b \in B$  with  $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ . Then clearly  $\ell(a \cap \kappa) \neq \ell(b \cap \kappa)$ . In fact  $\ell(a \cap \kappa) < \ell(b \cap \kappa)$  (since otherwise  $a \cap \kappa \notin S_{b \cap \kappa}$  and therefore  $\cup(a \cap \kappa) > \cup(b \cap \kappa)$ ). Hence  $b \cap \kappa \notin S_{a \cap \kappa}$ , so  $f(\cup(a \cap \kappa)) \subseteq b \cap \kappa$ .  $\square$

**Definition.** For  $A \subseteq P_\kappa(\lambda)$ , let

$$[A]_\kappa^2 = \{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\}$$

**Remark.**

$$[P_\kappa(\lambda)]_\kappa^2 = \{(\alpha, b) \in \kappa \times P_\kappa(\lambda) : \alpha < \cup(b \cap \kappa)\}.$$

**Definition.** For  $a, b \in P_\kappa(\lambda)$ , let  $a \prec b$  just in case  $a \subseteq b$  and  $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ .

**Definition.** For  $A \subseteq P_\kappa(\lambda)$ , let

$$[A]_{\prec}^2 = \{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } a \prec b\}.$$

**Remark.**  $[P_\kappa(\lambda)]_{\prec}^2 = [P_\kappa(\lambda)]_\kappa^2$ .

**THEOREM 9.4.** Suppose that  $\kappa$  is inaccessible and  $H$  is an ideal on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \text{cov}(\mathbf{M}_{\kappa, \kappa})$ , and let  $A \in H^+$ . Then there is  $C \in H^+ \cap P(A)$  such that  $[C]_\kappa^2 = [C]_{\prec}^2$ .

**Proof.** For  $\alpha < \kappa$ , set  $A_\alpha = \{a \in A : \cup(a \cap \kappa) = \alpha\}$ . By induction on  $\alpha < \kappa$ , we define  $c_k \in \{\phi\} \cup A_\alpha$  for  $k \in {}^\alpha 2$  as follows. Given  $k \in {}^\alpha 2$ , set

$$e_k = \bigcup \{c_{k \upharpoonright \beta} : \beta \in k^{-1}(\{1\})\}$$

and

$$Z_k = \{a \in A_\alpha : e_k \subseteq a\}.$$

If  $Z_k \neq \phi$ , let  $c_k$  be an arbitrary member of  $Z_k$ . Otherwise let  $c_k = \phi$ .

Set  $\nu = \text{cof}(H)$  and pick  $B_\xi \in H$  for  $\xi < \nu$  so that  $H = \bigcup_{\xi < \nu} P(B_\xi)$ . Let  $\xi < \nu$ . For  $\alpha < \kappa$ , let  $D_\xi^\alpha$  be the set of all  $s \in {}^{(\alpha+1)}2$  such that (i)  $s(\alpha) = 1$ , and (ii) there is  $a \in A_\alpha - B_\xi$  with the property that

$$(\forall \beta \in \alpha \cap s^{-1}(\{1\}))(\forall k \in {}^\beta 2) \quad c_k \subseteq a.$$

Then let  $D_\xi = \bigcup_{\alpha < \kappa} D_\xi^\alpha$  and  $U_\xi = \bigcup_{s \in D_\xi} O_s^\kappa$ . Let us prove that the open set  $U_\xi$  is dense. Thus let  $\gamma < \kappa$  and  $p \in {}^\gamma 2$ . Pick  $a \in (\bigcup_{\gamma \leq \delta < \kappa} A_\delta) - B_\xi$  so that

$$(\forall \beta \in p^{-1}(\{1\}))(\forall k \in {}^\beta 2) \quad c_k \subseteq a.$$

Set  $\alpha = \cup(a \cap \kappa)$  and define  $s \in {}^{(\alpha+1)}2$  by :  $s \upharpoonright \gamma = p$ ,  $s(\delta) = 0$  if  $\gamma \leq \delta < \alpha$ , and  $s(\alpha) = 1$ . It is immediate that  $s \in D_\xi^\alpha$ .

Select  $f \in \bigcap_{\xi < \nu} U_\xi$ . For each  $\xi < \nu$ , there is  $s_\xi \in D_\xi$  such that  $s_\xi \subset f$ . Let  $\alpha_\xi < \kappa$  be such that  $s_\xi \in D_\xi^{\alpha_\xi}$ . Set  $T = \{\alpha_\xi : \xi < \nu\}$  and define  $g \in {}^\kappa 2$  so that  $g^{-1}(\{1\}) = T$ . For  $\xi < \nu$ , set

$$d_\xi = \bigcup \{c_{g \upharpoonright \beta} : \beta \in T \cap \alpha_\xi\}$$

and

$$C_\xi = \{b \in A_{\alpha_\xi} : d_\xi \subseteq b\}.$$

Finally, let  $C = \bigcup_{\xi < \nu} C_\xi$ .

Let us verify that  $C$  is as desired. It is clear that  $C \subseteq A$ . Let  $\xi < \nu$ . There is  $a_\xi \in A_{\alpha_\xi} - B_\xi$  such that

$$(\forall \beta \in \alpha_\xi \cap s_\xi^{-1}(\{1\}))(\forall k \in {}^\beta 2) \quad c_k \subseteq a_\xi.$$

Put  $k_\xi = g \upharpoonright \alpha_\xi$ . Then  $a_\xi \in Z_{k_\xi}$  since  $s_\xi(\beta) = f(\beta) = 1$  for every  $\beta \in T \cap \alpha_\xi$ . It follows that  $c_{k_\xi} \in Z_{k_\xi}$ . It is immediate that  $Z_{k_\xi} = C_\xi$ . Thus we have shown that (a)  $C_\xi - B_\xi \neq \phi$  for every  $\xi < \nu$ , and (b)  $c_{g \upharpoonright \alpha_\xi} \in C_\xi$  for every  $\xi < \nu$ . It follows from (a) that  $C \in H^+$ , and from (b) that  $[C]_\kappa^2 = [C]_{\prec}^2$  since given  $\xi, \zeta < \nu$  with  $\alpha_\xi < \alpha_\zeta$ , we have  $c_{g \upharpoonright \alpha_\xi} \subseteq b$  for every  $b \in C_\zeta$ .  $\square$

**QUESTION.** Is the assumption that  $\kappa$  is inaccessible necessary in the statement of Theorem 9.4 ?

**Remark.** Suppose  $\kappa$  is inaccessible. Then by Theorems 9.1, 9.2, 9.4, 5.4 and 4.7,  $I_{\kappa, \lambda}$  is a weak  $\chi$ -point if and only if  $\lambda^{<\kappa} < \text{cov}(\mathbf{M}_{\kappa, \kappa})$  if and only if  $\{C : [C]_\kappa^2 = [C]_{\prec}^2\}$  is dense in  $(I_{\kappa, \lambda}^+, \subseteq)$ .



# 10. $H^+ \xrightarrow{\kappa} (H^+, \alpha)^2$

**Definition.** Let  $H$  be an ideal on  $P_\kappa(\lambda)$  and  $\alpha$  an ordinal.  $H^+ \xrightarrow{\kappa} (H^+, \alpha)^2$  means that given  $F : [P_\kappa(\lambda)]_\kappa^2 \rightarrow 2$  and  $A \in H^+$ , there is  $B \subseteq A$  such that either  $B \in H^+$  and  $F$  is identically 0 on  $[B]_\kappa^2$  or  $(B, \prec)$  has order type  $\alpha$  and  $F$  is identically 1 on  $[B]_\kappa^2$ .

In this section we show that  $H^+ \xrightarrow{\kappa} (H^+, \omega + 1)^2$  for every ideal  $H$  on  $P_\kappa(\lambda)$  with  $\text{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa, \kappa})$ .

**Definition.** Suppose that  $H$  is an ideal on  $P_\kappa(\lambda)$ ,  $A \in H^+$  and  $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$ . Then  $(H, A, F)$  is 0-good if there is  $D \in H^+ \cap P(A)$  such that  $\{b \in D : F(\cup(a \cap \kappa), b) = 1\} \in H$  for any  $a \in D$ .

The following is straightforward.

**LEMMA 10.1.** Suppose that  $(H, A, F)$  is 0-good, where  $H$  is an ideal on  $P_\kappa(\lambda)$  which is both a weak  $\pi$ -point and a weak  $\chi$ -point,  $A \in H^+$  and  $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$ . Then  $F$  is identically 0 on  $[C]_\kappa^2$  for some  $C \in H^+ \cap P(A)$ .

**Definition.** Given an ideal  $H$  on  $P_\kappa(\lambda)$  and  $B \in H^+$ , let  $M_{H, B}^d$  be the set of all  $Q \subseteq H^+ \cap P(B)$  such that (i) any two distinct members of  $Q$  are disjoint, and (ii) for every  $A \in H^+ \cap P(B)$ , there is  $C \in Q$  with  $A \cap C \in H^+$ .

**LEMMA 10.2.** Suppose that  $(H, A, F)$  is not 0-good, where  $H$  is an ideal on  $P_\kappa(\lambda)$ ,  $A \in H^+$  and  $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$ , and let  $B \in H^+ \cap P(A)$ . Then there exist  $Q_B \in M_{H, B}^d$  and  $\varphi_B : Q_B \rightarrow B$  such that (i)  $\varphi_B(D) \prec b$  and  $F(\cup(\varphi_B(D) \cap \kappa), b) = 1$  whenever  $b \in D \in Q_B$ , and (ii)  $\cup(\varphi_B(D) \cap \kappa) \neq \cup(\varphi_B(D') \cap \kappa)$  for any two distinct members  $D$  and  $D'$  of  $Q_B$ .

**Proof.** Set  $T = \{\cup(a \cap \kappa) : a \in B\}$  and define  $\psi : T \times (H^+ \cap P(B)) \rightarrow P(B)$  by  $\psi(\alpha, C) = \{b \in C : F(\alpha, b) = 1\}$ . Now using induction, define  $\eta \leq \kappa$  and  $\alpha_\delta \in T$  and  $B_\delta \in H^+ \cap P(B)$  for  $\delta < \eta$  so that :

$$(0) \quad \text{If } \delta < \eta, B - (\bigcup_{\xi < \delta} B_\xi) \in H^+,$$

$$\alpha_\delta = \text{least } \alpha \in T \text{ such that } \psi(\alpha, B - (\bigcup_{\xi < \delta} B_\xi)) \in H^+$$

$$\text{and } B_\delta = \psi(\alpha_\delta, B - (\bigcup_{\xi < \delta} B_\xi)).$$

$$(1) \quad \text{If } \eta < \kappa, B - (\bigcup_{\xi < \eta} B_\xi) \in H.$$

Notice that if  $\gamma < \delta < \eta$ , then

$$\psi(\alpha_\delta, B - (\bigcup_{\xi < \delta} B_\xi)) \subseteq \psi(\alpha_\delta, B - (\bigcup_{\zeta < \delta} B_\zeta))$$

and consequently  $\alpha_\gamma \leq \alpha_\delta$ . In fact  $\alpha_\gamma < \alpha_\delta$  as  $\psi(\alpha_\gamma, B - (\bigcup_{\xi < \delta} B_\xi)) = \emptyset$  (since  $(B - \bigcup_{\xi < \delta} B_\xi) \cap B_\gamma = \emptyset$ )

$$\text{and } B_\gamma = \{b \in B - (\bigcup_{\zeta < \gamma} B_\zeta) : F(\alpha_\gamma, b) = 1\}.$$

We claim that  $\{B_\delta : \delta < \eta\} \in M_{H,B}^d$ . Suppose otherwise. Then there exists  $E \in H^+ \cap P(B)$  such that  $E \cap B_\xi \in H$  for every  $\xi < \eta$ . Since

$$E - (\bigcup_{\xi < \delta} B_\xi) \in H^+ \cap P(B - (\bigcup_{\xi < \delta} B_\xi))$$

for every  $\delta < \kappa$ , we must have  $\eta = \kappa$ . Set

$$\beta = \text{least } \alpha \in T \text{ such that } \psi(\alpha, E) \in H^+.$$

Then for each  $\delta < \kappa$ ,

$$\psi(\beta, E) - (\bigcup_{\xi < \delta} B_\xi) \in H^+ \cap P(\psi(B, B - (\bigcup_{\xi < \delta} B_\xi)))$$

and therefore  $\beta \geq \alpha_\delta$ , which is a contradiction.

For each  $\delta < \eta$ , pick  $s_\delta \in B$  so that  $\cup(s_\delta \cap \kappa) = \alpha_\delta$ , and put

$$S_\delta = \{b \in B_\delta : s_\delta \cup (\alpha_\delta + 2) \subseteq b\}.$$

Finally, set  $Q_B = \{S_\delta : \delta < \eta\}$  and define  $\varphi_B : Q_B \rightarrow B$  by  $\varphi_B(S_\delta) = s_\delta$ . □

**LEMMA 10.3.** Suppose that  $H$  is an ideal on  $P_\kappa(\lambda)$  and  $A \in H^+$ . Suppose further that  $C \in H^+ \cap P(A)$  and  $Q_\alpha \in M_{H,A}^d$  for  $\alpha < \beta$ , where  $\beta$  is a limit ordinal with  $0 < \beta < \kappa$ . Then

$$\{a \in C : (\forall h \in \prod_{\alpha < \beta} Q_\alpha) \ a \notin \bigcap_{\alpha < \beta} h(\alpha)\} \in H.$$

**Proof.** It suffices to observe that for each  $a \in \bigcap_{\alpha < \beta} (C \cap (\cup Q_\alpha))$ , there is  $h \in \prod_{\alpha < \beta} Q_\alpha$  such that

$$a \in \bigcap_{\alpha < \beta} h(\alpha). \quad \square$$

**LEMMA 10.4.** Suppose that  $(H, A, F)$  is not 0-good, where  $H$  is an ideal on  $P_\kappa(\lambda)$ ,  $A \in H^+$  and  $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$ . Then :

- (i) There is  $C \subseteq A$  such that  $(C, \prec)$  has order type  $\omega + 1$  and  $F$  is identically 1 on  $[C]_\kappa^2$ .
- (ii) Suppose that  $\mathfrak{a}_H > \kappa$  and  $\theta$  is uncountable cardinal  $< \kappa$  such that  $\kappa \rightarrow (\kappa, \theta)^2$ . Then there is  $C \subseteq A$  such that  $(C, \prec)$  has order type  $\theta + 1$  and  $F$  is identically 1 on  $[C]_\kappa^2$ .

**Proof.** We prove (ii) and leave the proof of (i) to the reader. By Corollary 19.7 in [EHMáR], we have that  $\mu^\tau < \kappa$  whenever  $\mu$  and  $\tau$  are cardinals such that  $\theta \leq \mu < \kappa$  and  $0 < \tau < \theta$ . Using this and Lemmas 10.2 and 10.3, define  $R_\beta, Q_\beta \in \{W \in M_{H,A}^d : |W| < \kappa\}$  and  $\varphi_\beta : Q_\beta \rightarrow A$  for  $\beta < \theta$  by :

$$(0) \quad R_0 = \{A\} ;$$

$$(1) \quad Q_\beta = \bigcup_{B \in R_\beta} Q_B ;$$

$$(2) \quad R_{\beta+1} = Q_\beta ;$$

$$(3) \quad R_\beta = H^+ \cap \left\{ \bigcap_{\alpha < \beta} h(\alpha) : h \in \prod_{\alpha < \beta} Q_\alpha \right\} \text{ if } \beta \text{ is a limit ordinal } > 0 ;$$

$$(4) \quad \varphi_\beta = \bigcup_{B \in R_\beta} \varphi_B.$$

Select  $b \in \bigcap_{\beta < \theta} (\cup Q_\beta)$ . There must be  $k \in \prod_{\beta < \theta} Q_\beta$  such that  $b \in \bigcap_{\beta < \theta} k(\beta)$ . Then

$$C = \{\varphi_\beta(k(\beta)) : \beta < \theta\} \cup \{b\}$$

is as desired.  $\square$

**THEOREM 10.5.** Suppose  $\theta$  is an infinite cardinal  $< \kappa$  such that  $\kappa \rightarrow (\kappa, \theta)^2$ . Then  $H^+ \xrightarrow{\kappa} (H^+, \theta + 1)^2$  for every ideal  $H$  on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa, \kappa})$ .

**Proof.** Let  $H$  be an ideal on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa, \kappa})$ . Then  $H$  is a weak  $\chi$ -point by Theorem 9.1. Moreover,  $H$  is a weak  $\pi$ -point and  $\mathfrak{a}_H > \kappa$  by Theorem 8.4 since  $\mathbf{cov}(\mathbf{M}_{\kappa, \kappa}) \leq \mathfrak{d}_{\kappa, \lambda}^\kappa$  by Proposition 4.1. Hence,  $H^+ \xrightarrow{\kappa} (H^+, \theta + 1)^2$  by Lemmas 10.1 and 10.4.  $\square$

## 11. $H^+ \xrightarrow[\kappa]{} (H^+, \alpha)^2$

**Definition.** For  $A \subseteq P_\kappa(\lambda)$ , let

$$[A]_{\kappa, \kappa}^2 = \{(\cup(a \cap \kappa), \cup(b \cap \kappa)) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\}$$

**Remark.**  $[P_\kappa(\lambda)]_{\kappa, \kappa}^2 = [\kappa]^2$ .

**Definition.** Let  $H$  be an ideal on  $P_\kappa(\lambda)$  and  $\alpha$  an ordinal.  $H^+ \xrightarrow[\kappa]{} (H^+, \alpha)^2$  means that given  $F : [P_\kappa(\lambda)]_{\kappa, \kappa}^2 \rightarrow 2$  and  $A \in H^+$ , there is  $B \subseteq A$  such that either  $B \in H^+$  and  $F$  is identically 0 on  $[B]_{\kappa, \kappa}^2$ , or  $(B, <)$  has order type  $\alpha$  and  $F$  is identically 1 on  $[B]_{\kappa, \kappa}^2$ .

We will show that  $H^+ \xrightarrow[\kappa]{} (H^+, \omega + 1)^2$  for every ideal  $H$  on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathbf{non}_\kappa$  (weakly selective).

**Definition.** For an ideal  $H$  on  $P_\kappa(\lambda)$ ,  $J_H = \{B \subseteq \kappa : U_B \in H\}$ , where

$$U_B = \{a \in P_\kappa(\lambda) : \cup(a \cap \kappa) \in B\}.$$

**LEMMA 11.1.** Let  $H$  be an ideal on  $P_\kappa(\lambda)$ . Then  $J_H$  is an ideal on  $\kappa$ . Moreover,  $\text{cof}(J_H) \leq \text{cof}(H)$ .

**Proof.** It is simple to see that (a)  $U_\kappa = P_\kappa(\lambda)$ , (b)  $U_{\cup \mathfrak{B}} \subseteq \bigcup_{B \in \mathfrak{B}} U_B$  for  $\mathfrak{B} \subseteq P(\kappa)$ , (c)  $U_C \subseteq U_B$  if  $C \subseteq B \subseteq \kappa$ , and (d)  $U_B \in I_{\kappa, \lambda}$  for every  $B \subseteq \kappa$  with  $|B| = 1$ . The first assertion immediately follows.

For  $C \subseteq P_\kappa(\lambda)$ , let  $Y_C$  be the set of all  $\delta \in \kappa$  such that

$$\{a \in P_\kappa(\lambda) : \cup(a \cap \kappa) = \delta\} \subseteq C.$$

If  $C \in H$ , then  $Y_C \in J_H$  since  $U_{Y_C} \subseteq C$ . Moreover if  $B \subseteq \kappa$  and  $U_B \subseteq C \subseteq P_\kappa(\lambda)$ , then  $B \subseteq Y_C$ . Hence  $\text{cof}(J_H) \leq \text{cof}(H)$ .  $\square$

**Remark.** Let  $H$  be an ideal on  $P_\kappa(\lambda)$ . Then

$$\{\cup(a \cap \kappa) : a \in A\} \in (J_{H|A})^+$$

for every  $A \in H^+$ .

The following is readily checked.

**LEMMA 11.2.** *Given an ideal  $H$  on  $P_\kappa(\lambda)$ , the following are equivalent :*

- (i)  $J_H$  is a local  $Q$ -point.
- (ii) For every  $g \in {}^\kappa\kappa$ , there is  $B \in H^+$  such that  $g(\cup(a \cap \kappa)) < \cup(b \cap \kappa)$  for all  $a, b \in B$  with  $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ .

Suppose  $\kappa$  is a limit cardinal. If  $\kappa^+ < \mathbf{non}_\kappa(\text{weak } Q\text{-point})$ , then by Lemma 11.1  $J_{I_{\kappa, \kappa^+}|A}$  is a local  $Q$ -point for every  $A \in I_{\kappa, \kappa^+}^+$ . The following shows that this implication can be reversed.

**PROPOSITION 11.3.** *Suppose that  $\kappa$  is a limit cardinal and  $J_{I_{\kappa, \lambda}|A}$  is a local  $Q$ -point for every  $A \in I_{\kappa, \lambda}^+$ . Then  $\sigma < \overline{\mathbf{non}}_\kappa(\text{weak } Q\text{-point})$  for every  $\sigma \in \mathcal{K}(\kappa, \lambda)$ .*

**Proof.** Suppose that  $J$  is an ideal on  $\kappa$  and  $T \subseteq P_\kappa(\lambda - \kappa)$  is such that  $\overline{\text{cof}}(J) \leq |T|$  and  $|T \cap P(a)| < \kappa$  for every  $a \in P_\kappa(\lambda)$ . Select  $B_d \in J$  for  $d \in T$  so that for every  $D \in J$ , there is  $u \in P_\kappa(T) - \{\phi\}$  with  $D \subseteq \bigcup_{d \in u} B_d$ . Let  $A$  be the set of all  $a \in P_\kappa(\lambda)$  such that  $\cup(a \cap \kappa) \notin B_d$  for every  $d \in T \cap P(a - \kappa)$ .

It is simple to see that  $A \in I_{\kappa, \lambda}^+$ . Now fix  $g \in {}^\kappa\kappa$ . By Lemma 11.2, there is  $C \in (I_{\kappa, \lambda} | A)^+$  such that  $g(\cup(a \cap \kappa)) < \cup(b \cap \kappa)$  for all  $a, b \in C$  with  $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ . Set

$$D = \{\cup(a \cap \kappa) : a \in C \cap A\}.$$

Then  $D \in J^+$ . Moreover  $g(\alpha) < \beta$  for all  $\alpha, \beta \in D$  with  $\alpha < \beta$ . Hence  $J$  is a local  $Q$ -point.  $\square$

**THEOREM 11.4.** *Suppose that  $\theta$  is an infinite cardinal  $< \kappa$  such that  $\kappa \rightarrow (\kappa, \theta)^2$ , and  $H$  is an ideal on  $P_\kappa(\lambda)$  with  $\text{cof}(H) < \mathbf{non}_\kappa(\text{weakly selective})$ . Then  $H^+ \xrightarrow[\kappa]{\kappa} (H^+, \theta + 1)^2$ .*

**Proof.** Fix  $G : \kappa \times \kappa \rightarrow 2$  and  $A \in H^+$ . Define  $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$  by  $F(\alpha, b) = G(\alpha, \cup(b \cap \kappa))$ .

First suppose  $(H, A, F)$  is 0-good. Pick  $D \in H^+ \cap P(A)$  so that

$$\{b \in D : F(\cup(a \cap \kappa), b) = 1\} \in H$$

for any  $a \in D$ . Set  $B_\alpha = \{\delta < \kappa : G(\alpha, \delta) = 1\}$  for  $\alpha < \kappa$ . Then  $B_{\cup(a \cap \kappa)} \in J_{H|D}$  for every  $a \in D$  since

$$D \cap U_{B_{\cup(a \cap \kappa)}} = \{b \in D : G(\cup(a \cap \kappa), \cup(b \cap \kappa)) = 1\} = \{b \in D : F(\cup(a \cap \kappa), b) = 1\}.$$

By Lemma 11.1  $\text{cof}(J_{H|D}) < \mathbf{non}_\kappa(\text{weak } P\text{-point})$  so there is  $G \in (J_{H|D})^+$  such that  $|G \cap B_{\cup(a \cap \kappa)}| < \kappa$  for every  $a \in D$ . Notice that  $D \cap U_G \in H^+$ . Select  $g \in {}^\kappa\kappa$  so that  $\cup(b \cap \kappa) \notin B_{\cup(a \cap \kappa)}$  for all  $a, b \in D \cap U_G$  such that  $g(\cup(a \cap \kappa)) < \cup(b \cap \kappa)$ . By Lemma 11.1

$$\text{cof}(J_{H|(D \cap U_G)}) < \mathbf{non}_\kappa(\text{weak } Q\text{-point})$$

and hence by Lemma 11.2 there is  $R \in (H | (D \cap U_G))^+$  such that  $g(\cup(a \cap \kappa)) < \cup(b \cap \kappa)$  for all  $a, b \in R$  with  $\cup(a \cap \kappa) < \cup(b \cap \kappa)$ . Then  $R \cap D \cap U_G \in H^+ \cap P(A)$  and moreover  $F$  is identically 0 on  $[R \cap D \cap U_G]_{\kappa, \kappa}^2$ .

Finally, suppose  $(H, A, F)$  is not 0-good. Since  $\mathfrak{a}_H > \kappa$  by Theorems 2.2 and 8.4, there is by Lemma 10.4  $C \subseteq A$  such that  $(C, \prec)$  has order type  $\theta + 1$  and  $F$  is identically 1 on  $[C]_{\kappa}^2$ . It is immediate that  $G$  is constantly 1 on  $[C]_{\kappa, \kappa}^2$ .  $\square$

**Remark.** Suppose  $\kappa$  is a successor cardinal. Then by Theorem 11.4  $\kappa^+ < \mathfrak{d}_\kappa$  implies that  $I_{\kappa, \kappa^+}^+ \xrightarrow[\kappa]{\kappa} (I_{\kappa, \kappa^+}^+, \theta + 1)^2$  for every cardinal  $\theta \geq 2$  such that  $\kappa \rightarrow (\kappa, \theta)^2$ . Conversely, it will be shown in the next section that  $I_{\kappa, \kappa^+}^+ \xrightarrow[\kappa]{\kappa} (I_{\kappa, \kappa^+}^+, 3)^2$  implies that  $\kappa^+ < \mathfrak{d}_\kappa$ .

## 12. $H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$

**Definition.** Given an ideal  $H$  on  $P_\kappa(\lambda)$  and an ordinal  $\alpha$ ,  $H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$  means that for all  $F : [P_\kappa(\lambda)]_{\kappa, \kappa}^2 \rightarrow 2$  and  $A \in H^+$ , there is  $B \subseteq A$  such that either  $B \in H^+$  and  $F$  is identically 0 on  $[B]_{\kappa, \kappa}^2$ , or  $\{\cup(a \cap \kappa) : a \in B\}$  has order type  $\alpha$  and  $F$  is identically 1 on  $[B]_{\kappa, \kappa}^2$ .

**Remark.**  $H^+ \xrightarrow[\kappa]{\kappa} (H^+, \alpha)^2 \Rightarrow H^+ \xrightarrow[\kappa]{\kappa} (H^+, \alpha)^2 \Rightarrow H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2 \Rightarrow \kappa \rightarrow (\kappa, \alpha)^2$ .

We will prove that  $I_{\kappa, \kappa}^+ \xrightarrow[\kappa]{\kappa} (I_{\kappa, \kappa^+}^+; \alpha)^2$  if and only if  $\kappa^+ < \mathbf{non}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$ .

**THEOREM 12.1.** Suppose that  $3 \leq \alpha \leq \kappa$  and  $H$  is an ideal on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathbf{non}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$ . Then  $H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$ .

**Proof.** By Lemma 11.1,  $(J_{H|A})^+ \rightarrow ((J_{H|A})^+, \alpha)^2$  for every  $A \in H^+$ . The desired conclusion easily follows.  $\square$

**THEOREM 12.2.** Suppose that  $3 \leq \alpha \leq \kappa$  and  $I_{\kappa, \lambda}^+ \xrightarrow[\kappa]{\kappa} (I_{\kappa, \lambda}^+; \alpha)^2$ . Then  $\sigma < \overline{\mathbf{non}}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$  for every  $\sigma \in \mathcal{K}(\kappa, \lambda)$ .

**Proof.** The proof is an easy modification of that of Proposition 11.3.  $\square$

**Remark.** Suppose that  $\kappa$  is inaccessible and  $3 \leq \alpha \leq \kappa$ . Then by Theorems 12.1 and 12.2,  $I_{\kappa, \lambda}^+ \xrightarrow[\kappa]{\kappa} (I_{\kappa, \lambda}^+; \alpha)^2$  if and only if  $\lambda^{<\kappa} < \mathbf{non}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$ .

Let us finally observe that for  $3 \leq \alpha \leq \kappa$ , there always exists an ideal  $H$  on  $P_\kappa(\lambda)$  of the least possible cofinality such that  $H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$ :

**PROPOSITION 12.3.** Given  $3 \leq \alpha \leq \kappa$ , there is an ideal  $H$  on  $P_\kappa(\lambda)$  such that (a)  $H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$ , (b)  $\text{cof}(H) = u(\kappa, \lambda) \cdot \mathbf{non}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$ , and (c)  $\overline{\text{cof}}(H) \leq \lambda \cdot \overline{\mathbf{non}}_\kappa(J^+ \rightarrow (J^+, \alpha)^2)$ .

**Proof.** Argue as for Lemma 5.1 of [M2].  $\square$

### 13. $H^+ \xrightarrow{\kappa} (H^+)^2$

**Definition.** Given an ideal  $H$  on  $P_\kappa(\lambda)$ ,  $H^+ \xrightarrow{\kappa} (H^+)^2$  (respectively,  $H^+ \xrightarrow[\kappa]{\kappa} (H^+)^2$ ) means that for all  $F : [P_\kappa(\lambda)]_\kappa^2 \rightarrow 2$  (respectively,  $F : [P_\kappa(\lambda)]_{\kappa,\kappa}^2 \rightarrow 2$ ) and  $A \in H^+$ , there is  $B \in H^+ \cap P(A)$  such that  $F$  is constant on  $[B]_\kappa^2$  (respectively,  $[B]_{\kappa,\kappa}^2$ ).

**THEOREM 13.1.** Suppose  $\kappa$  is weakly compact. Then  $H^+ \xrightarrow{\kappa} (H^+)^2$  for every ideal  $H$  on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ .

**Proof.** Suppose that  $H$  is an ideal on  $P_\kappa(\lambda)$  with  $\text{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ ,  $F : \kappa \times P_\kappa(\lambda) \rightarrow 2$  and  $A \in H^+$ . Then  $\text{cof}(H) < \mathfrak{d}_{\kappa,\lambda}^\kappa$  by Proposition 4.1 and therefore by a result of [M5] there are  $B \in H^+ \cap P(A)$  and  $i < 2$  such that

$$\{b \in B : F(\cup(a \cap \kappa), b) \neq i\} \in I_{\kappa,\lambda}$$

for every  $a \in B$ . Since  $H$  is a weak  $\chi$ -point by Theorem 9.1, there is  $C \in H^+ \cap P(B)$  such that  $F$  takes the constant value  $i$  on  $[C]_\kappa^2$ .  $\square$

**Remark.** It follows from Theorem 6.5 (ii) and Theorem 15.1 (below) that if  $\kappa$  is weakly compact, then  $H^+ \xrightarrow[\kappa]{\kappa} (H^+)^2$  for every ideal  $H$  on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathbf{non}_\kappa$  (weakly selective).

**COROLLARY 13.2.** The following are equivalent :

- (i)  $\kappa$  is weakly compact and  $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ .
- (ii)  $I_{\kappa,\lambda}^+ \xrightarrow{\kappa} (I_{\kappa,\lambda}^+)^2$ .
- (iii)  $I_{\kappa,\lambda}^+ \xrightarrow[\kappa]{\kappa} (I_{\kappa,\lambda}^+; \kappa)^2$ .

**Proof.** (i)  $\rightarrow$  (ii) : By Theorem 13.1.

(ii)  $\rightarrow$  (iii) : Trivial.

(iii)  $\rightarrow$  (i) : By Theorems 12.2, 6.5 (i), 6.1 (iii), 5.4 and 4.7.  $\square$

### 14. $H^+ \xrightarrow{\kappa} [H^+]_\rho^2$

**Definition.** Given a cardinal  $\rho$  with  $2 \leq \rho \leq \lambda^{<\kappa}$  and an ideal  $H$  on  $P_\kappa(\lambda)$ ,  $H^+ \xrightarrow{\kappa} [H^+]_\rho^2$  means that for all  $F : [P_\kappa(\lambda)]_\kappa^2 \rightarrow \rho$  and  $A \in H^+$ , there is  $B \in H^+ \cap P(A)$  such that  $F''[B]_\kappa^2 \neq \rho$ .

**THEOREM 14.1.** Suppose that  $\kappa$  is a limit cardinal and  $H$  is an ideal on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ . Then  $H^+ \xrightarrow{\kappa} [H^+]_{\kappa^+}^2$ .

**Proof.** Fix  $F : \kappa \times P_\kappa(\lambda) \rightarrow \kappa^+$  and  $A \in H^+$ . Since  $\text{cof}(H) < \mathfrak{d}_{\kappa,\lambda}^\kappa$  by Proposition 4.1, there are  $B \in H^+ \cap P(A)$  and  $\xi \in \kappa^+$  such that  $\{b \in B : F(\cup(a \cap \kappa), b) = \xi\} \in I_{\kappa,\lambda}$  for every  $a \in B$  ([M5]). Now  $H$  is a weak  $\chi$ -point by Theorem 9.1 and so  $\xi \notin F''[C]_\kappa^2$  for some  $C \in H^+ \cap P(B)$ .  $\square$

Let us now show that  $I_{\kappa,\lambda}^+ \not\xrightarrow{\kappa} [I_{\kappa,\lambda}^+]_\lambda^2$  if  $\lambda \geq \bar{\mathfrak{d}}_\kappa$ . We will need some definitions.

**Definition.** Given  $f \in \prod_{\alpha \in \kappa} (\kappa - \alpha)$ , we define  $\tilde{f} \in {}^\kappa \kappa$  by stipulating that

- (i)  $\tilde{f}(0) = 0$  ;
- (ii)  $\tilde{f}(\xi + 1) = f(\tilde{f}(\xi)) + 1$  ;
- (iii)  $\tilde{f}(\xi) = \bigcup_{\zeta < \xi} \tilde{f}(\zeta)$  if  $\xi$  is a limit ordinal  $> 0$ .

**Remark.**  $\tilde{f}$  is a strictly increasing function.

**Remark.** If  $g \in {}^\kappa\kappa$  is a strictly increasing function such that  $g(\alpha) \leq f(\alpha)$  for all  $\alpha < \kappa$ , then  $g(\tilde{f}(\xi)) \in [\tilde{f}(\xi), \tilde{f}(\xi + 1))$  for every  $\xi < \kappa$ .

**Definition.** Given  $f \in \prod_{\alpha \in \kappa} (\kappa - \alpha)$  and a cardinal  $\tau \in (0, \kappa)$ , we define  $c_{f,\tau} : \tilde{f}(\tau) \rightarrow \tau$  by stipulating that  $c_{f,\tau}$  takes the constant value  $\xi$  on  $[\tilde{f}(\xi), \tilde{f}(\xi + 1))$ .

**Definition.** Suppose that  $T \subseteq P_\kappa(\lambda - \kappa)$  is such that (a)  $|T| \geq \bar{\mathfrak{d}}_\kappa$ , and (b)  $|T \cap P(a)| < \kappa$  for every  $a \in P_\kappa(\lambda)$ .

Let  $\psi_T : T \rightarrow {}^\kappa\kappa$  be such that given  $g \in {}^\kappa\kappa$ , there is  $u \in P_\kappa(T) - \{\phi\}$  such that

$$g(\alpha) \leq \bigcup_{d \in u} (\psi_T(d))(\alpha)$$

for all  $\alpha < \kappa$ .

For  $e \in P_\kappa(\lambda - \kappa)$ , let  $\tau_{T,e} = |T \cap P(e)|$  and select a bijection  $k_{T,e} : \tau_{T,e} \rightarrow T \cap P(e)$ .

Also, define  $f_{T,e} \in {}^\kappa\kappa$  by

$$f_{T,e}(\alpha) = \max(\alpha, \bigcup_{d \in T \cap P(e)} (\psi_T(d))(\alpha)).$$

Finally, let  $A_T$  be the set of all  $a \in P_\kappa(\lambda)$  such that (i)  $T \cap P(a - \kappa) \neq \phi$ , and (ii)  $\cup(a \cap \kappa) \geq \tilde{f}_{T,a-\kappa}(\tau_{T,a-\kappa})$ .

**Remark.**  $A_T \in I_{\kappa,\lambda}^+$ .

**THEOREM 14.2.** Suppose that  $\rho \in \mathcal{K}(\kappa, \lambda)$  and  $\rho \geq \bar{\mathfrak{d}}_\kappa$ . Then  $I_{\kappa,\lambda}^+ \not\stackrel{\kappa}{\rightarrow} [I_{\kappa,\lambda}^+]_\rho^2$ .

**Proof.** Select  $T \subseteq P_\kappa(\lambda - \kappa)$  so that  $|T| = \rho$  and  $|T \cap P(a)| < \kappa$  for every  $a \in P_\kappa(\lambda)$ . We define a partial function  $F$  from  $\kappa \times A_T$  to  $T$  by stipulating that

$$F(\beta, a) = k_{T,a-\kappa}(c_{f_{T,a-\kappa}, \tau_{T,a-\kappa}}(\beta))$$

if  $a \in A_T$  and  $\beta < \tilde{f}_{T,a-\kappa}(\tau_{T,a-\kappa})$ .

Now fix  $B \in I_{\kappa,\lambda}^+ \cap P(A_T)$  and  $x \in T$ . Let  $g \in {}^\kappa\kappa$  be the increasing enumeration of the elements of the set  $\{\cup(b \cap \kappa) : b \in B\}$ . Select  $u \in P_\kappa(T) - \{\phi\}$  so that  $g(\alpha) \leq \bigcup_{d \in u} (\psi_T(d))(\alpha)$  for all  $\alpha < \kappa$ . Now pick  $a \in B$  so that  $x \cup (\cup u) \subseteq a$ . Notice that  $g(\alpha) \leq f_{T,a-\kappa}(\alpha)$  for every  $\alpha \in \kappa$ . Let  $\xi \in \tau_{T,a-\kappa}$  be such that  $k_{T,a-\kappa}(\xi) = x$ . Then

$$\tilde{f}_{T,a-\kappa}(\xi) \leq g(\tilde{f}_{T,a-\kappa}(\xi)) < \tilde{f}_{T,a-\kappa}(\xi + 1) \leq \tilde{f}_{T,a-\kappa}(\tau_{T,a-\kappa}) \leq \cup(a \cap \kappa).$$

Moreover,

$$F(g(\tilde{f}_{T,a-\kappa}(\xi)), a) = k_{T,a-\kappa}(\xi) = x.$$

since

$$c_{f_{T,a-\kappa}, \tau_{T,a-\kappa}}(g(\tilde{f}_{T,a-\kappa}(\xi))) = \xi$$

□

## 15. $H^+ \xrightarrow[\kappa]{\kappa} [H^+]_\rho^2$

**Definition.** Given a cardinal  $\rho \in [2, \kappa]$  and an ideal  $H$  on  $P_\kappa(\lambda)$ ,  $H^+ \xrightarrow[\kappa]{\kappa} [H^+]_\rho^2$  means that for all  $F : [P_\kappa(\lambda)]_{\kappa, \kappa}^2 \rightarrow \rho$  and  $A \in H^+$ , there is  $B \in H^+ \cap P(A)$  such that  $F''[B]_{\kappa, \kappa}^2 \neq \rho$ .

**Remark.**  $\kappa \not\rightarrow [\kappa]_\rho^2 \Rightarrow H^+ \xrightarrow[\kappa]{\kappa} [H^+]_\rho^2 \Rightarrow H^+ \xrightarrow[\kappa]{\kappa} [H^+]_\rho^2$ .

The following result shows that  $I_{\kappa, \kappa^+}^+ \xrightarrow[\kappa]{\kappa} [I_{\kappa, \kappa^+}^+]_\rho^2$  if and only if  $\kappa^+ < \mathbf{non}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$ .

**THEOREM 15.1.** Let  $\rho$  be a cardinal with  $2 \leq \rho \leq \kappa$ . Then :

- (i)  $H^+ \xrightarrow[\kappa]{\kappa} [H^+]_\rho^2$  for every ideal  $H$  on  $P_\kappa(\lambda)$  such that  $\text{cof}(H) < \mathbf{non}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$ .
- (ii) If  $I_{\kappa, \lambda}^+ \xrightarrow[\kappa]{\kappa} [I_{\kappa, \lambda}^+]_\rho^2$ , then  $\sigma < \overline{\mathbf{non}}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$  for every  $\sigma \in \mathcal{K}(\kappa, \lambda)$ .

**Proof.** (i) : Use Lemma 11.1.

(ii) : Argue as for Proposition 11.3. □

**Remark.** Thus assuming  $\kappa$  is inaccessible,  $I_{\kappa, \lambda}^+ \xrightarrow[\kappa]{\kappa} [I_{\kappa, \lambda}^+]_\rho^2$  if and only if  $\lambda^{<\kappa} < \mathbf{non}_\kappa(J^+ \rightarrow [J^+]_\rho^2)$ .

Finally, we show that if  $\lambda \geq \overline{\mathfrak{d}}_\kappa$  and  $\kappa$  is a limit cardinal such that  $2^{<\kappa} = \kappa$ , then  $I_{\kappa, \lambda}^+ \xrightarrow[\kappa]{\kappa} [I_{\kappa, \lambda}^+]_\kappa^2$ .

**THEOREM 15.2.** Suppose that (a)  $\kappa$  is a limit cardinal such that  $2^{<\kappa} = \kappa$ , and (b) either  $\lambda > \overline{\mathfrak{d}}_\kappa$ , or  $\overline{\mathfrak{d}}_\kappa \in \mathcal{K}(\kappa, \lambda)$ . Then  $I_{\kappa, \lambda}^+ \xrightarrow[\kappa]{\kappa} [I_{\kappa, \lambda}^+]_\kappa^2$ .

**Proof.** Select  $T \subseteq P_\kappa(\lambda - \kappa)$  so that  $|T| = \lambda \cdot \overline{\mathfrak{d}}_\kappa$  and  $|T \cap P(a)| < \kappa$  for every  $a \in P_\kappa(\lambda)$ . Also, select  $\chi : \kappa \rightarrow \bigcup_{\gamma < \kappa} \gamma \kappa$  so that  $|\chi^{-1}(\{z\})| = \kappa$  for every  $z \in \bigcup_{\gamma < \kappa} \gamma \kappa$ . Now let  $A$  be the set of all  $a \in A_T$  such that

$$\chi(\cup(a \cap \kappa)) = c_{f_{T, a-\kappa}, \tau_{T, a-\kappa}}.$$

Notice that  $A \in I_{\kappa, \lambda}^+$ . We define a partial function  $F$  from  $\kappa \times \kappa$  to  $\kappa$  by stipulating that  $F(\delta, \eta) = (\chi(\eta))(\delta)$  if  $\eta \in \kappa$  and  $\delta \in \text{dom}(\chi(\eta))$ .

Now fix  $B \in I_{\kappa, \lambda}^+ \cap P(A)$  and  $\xi \in \kappa$ . Let  $g \in {}^\kappa \kappa$  be the increasing enumeration of the elements of the set  $\{\cup(b \cap \kappa) : b \in B\}$ . Select  $u \in P_\kappa(T) - \{\emptyset\}$  so that  $g(\alpha) \leq \bigcup_{d \in u} (\psi_T(d))(\alpha)$  for all  $\alpha < \kappa$ . Pick  $a \in B$  so that  $\cup u \subseteq a$  and  $|T \cap P(a)| > \xi$ . Then

$$g(\tilde{f}_{T, a-\kappa}(\xi)) < \cup(a \cap \kappa)$$

and

$$\xi = c_{f_{T, a-\kappa}, \tau_{T, a-\kappa}}(g(\tilde{f}_{T, a-\kappa}(\xi))) = (\chi(\cup(a \cap \kappa)))(g(\tilde{f}_{T, a-\kappa}(\xi))) = F(g(\tilde{f}_{T, a-\kappa}(\xi)), \cup(a \cap \kappa)).$$

□

**Remark.** Theorems 14.2, 15.1 and 15.2 (as well as e.g. Theorems 9.2, 9.4, 12.1 and 12.2, Propositions 9.3 and 11.3 and Corollary 13.2) are also true for  $\kappa = \omega$ . This gives (a)  $\mathfrak{d} \geq \mathbf{non}_\omega(J^+ \rightarrow [J^+]_\omega^2)$ , and (b) if  $\lambda \geq \mathfrak{d}$ , then  $I_{\omega, \lambda}^+ \xrightarrow[\omega]{\omega} [I_{\omega, \lambda}^+]_\lambda^2$  and  $I_{\omega, \lambda}^+ \xrightarrow[\omega]{\omega} [I_{\omega, \lambda}^+]_\omega^2$ .



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